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On an Inverse Problem in Group Analysis of PDE's: Lie–Remarkable Equations

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Abstract

Within the framework of inverse Lie problems we give some non–trivial examples of Lie–remarkable equations, *i.e.*, classes of partial differential equations that are in one–to–one correspondence with their Lie point symmetries. In particular, we prove that the second order Monge-Ampère equation in two independent variables is Lie–remarkable. The same property is shared by some classes of second order Monge-Ampère equations involving more than two independent variables, as well as by some classes of higher order Monge-Ampère equations in two independent variables. In closing, also the minimal surface equation in $\mathbb{R}^3$ is considered.

1 Introduction

Lie group analysis (see Refs. [1, 2, 3, 4, 5, 6, 7]) is a formidable tool for investigating differential equations in a general framework without using *ad hoc* methods: it may be used for determining the admitted symmetries useful for finding invariant solutions to differential equations, reducing the order of ordinary differential equations, transforming differential equations in more convenient forms [4, 8], etc.

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Roughly speaking, in dealing with Lie group analysis of partial differential equations (PDE's), either a direct problem or an inverse one may be considered. In the direct problem, starting with a system of PDE's

$$\widetilde{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(k)}\right) = 0, \tag{1}$$

where $\widetilde{\Delta}$ is an assigned function of the independent variables $\mathbf{x} \in \mathbb{R}^n$, the dependent variables $\mathbf{u} \in \mathbb{R}^m$, and $\mathbf{u}^{(k)}$ (the set of all partial derivatives of the $\mathbf{u}$'s with respect to the $\mathbf{x}$'s up to the order k), one is interested to find the admitted group of Lie symmetries.

Let us consider a one-parameter (ϵ) Lie group of point transformations

$$\mathbf{x}^\star = \mathbf{X}(\mathbf{x}, \mathbf{u}; \epsilon), \qquad \mathbf{u}^\star = \mathbf{U}(\mathbf{x}, \mathbf{u}; \epsilon), \tag{2}$$

(the transformation (2) for $\epsilon = 0$ reduces to identity).

The system (1) is invariant with respect to (2) if

$$\widetilde{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(k)}\right) = \widetilde{\Delta}\left(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{u}^{\star(k)}\right) = 0. \tag{3}$$

By expanding transformation (2) around $\epsilon = 0$, one gets the infinitesimal transformation

$$\begin{aligned}
\mathbf{x}^\star &= \mathbf{x} + \frac{d}{d\epsilon}\mathbf{X}(\mathbf{x}, \mathbf{u}; \epsilon)\Big|_{\epsilon=0} + O(\epsilon^2) = \mathbf{x} + \epsilon\boldsymbol{\xi}(\mathbf{x}, \mathbf{u}) + O(\epsilon^2), \\
\mathbf{u}^\star &= \mathbf{u} + \frac{d}{d\epsilon}\mathbf{U}(\mathbf{x}, \mathbf{u}; \epsilon)\Big|_{\epsilon=0} + O(\epsilon^2) = \mathbf{u} + \epsilon\boldsymbol{\eta}(\mathbf{x}, \mathbf{u}) + O(\epsilon^2),
\end{aligned}$$

to which it corresponds the infinitesimal operator (vector field)

$$\Xi = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i} + \sum_{A=1}^{m} \eta_A \frac{\partial}{\partial u_A}. \tag{4}$$

The straightforward Lie algorithm[3, 4] allows us to derive the general form of infinitesimal generators ξ_i and η_A of the Lie group admitted by (1) by solving an overdetermined system of linear PDE's arising from

$$\frac{d}{d\epsilon}\widetilde{\Delta}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{u}^{\star(k)})\Big|_{\epsilon=0, \widetilde{\Delta}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(k)})=0} = \Xi^{(k)}\widetilde{\Delta}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(k)})\Big|_{\widetilde{\Delta}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(k)})=0} = 0,$$

where $\Xi^{(k)}$ is the k–th prolongation of Ξ up to the order k; $\Xi^{(k)}$ is obtained by using some recurrence relations accounting for the transformations of derivatives. The integration of the determining equations gives the infinitesimal operators Ξ_i admitted by the system (1); these vector fields span a Lie algebra that can be finite or infinite dimensional.

The above constructions admit a standard geometrical interpretation. Eq. (1) can be seen as a submanifold in the space with coordinates $(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(k)})$ (the jet space[3]); then the operator $\Xi^{(k)}$ is just a vector field on the jet space, and is a symmetry if it is tangent to the submanifold (1).

On the contrary, in the inverse problem one chooses a Lie group of symmetries and determines the most general system (having an assigned structure) admitting it[9].

By imposing the invariance of an unspecified system of partial differential equations with respect to an assigned Lie group of symmetries one obtains a system of linear partial differential equations whose solution leads to consider the differential invariants of the Lie group[10, 5].

1 Definition. (see Ref. [11]) The function

$$I(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(k)})$$

is called a differential invariant of order k for the Lie algebra $\mathcal{L}$ spanned by the vector fields Ξ_i $(i = 1, \ldots, p)$ if

$$\Xi_i^{(k)} I(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(k)}) = \lambda_i(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(k)}) I(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(k)}), \qquad (i = 1, \ldots, p),$$

where $\lambda_i(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(k)})$ are some functions of the indicated arguments. If all the functions λ_i are vanishing, I is said an absolute differential invariant, whereas if some function $\lambda_i \neq 0$, I is said a relative differential invariant. $\square$

When the inverse Lie problem is considered, a maximal set of functionally independent differential invariants for the Lie algebra $\mathcal{L}$ is required[5]. The most general equation left invariant with respect to $\mathcal{L}$ will be given as an arbitrary function of the differential invariants. Within this context an interesting question may arise whether there exist non-trivial equations which are in one-to-one correspondence with their invariance groups (see Refs. [12, 13]). To this purpose, let us give the following definition[14].

2 Definition. Suppose we have a (system of) equation(s)

$$\widetilde{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(k)}\right) = 0, \tag{5}$$

with $\widetilde{\Delta}$ assigned function of its arguments, and determine the infinitesimal operators of its Lie symmetries, say

$$\Xi_1, \quad \Xi_2, \quad \ldots, \quad \Xi_p. \tag{6}$$

In general, if we consider a general system of PDE's

$$\Delta\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(k)}\right) = 0, \tag{7}$$

with Δ unspecified function of its arguments, and, by requiring that (7) has the Lie point symmetries (6), we find that

$$\Delta \equiv \widetilde{\Delta}, \tag{8}$$

then we call (5) a Lie–remarkable (system of) equation(s). $\square$

In Ref. [15] this definition has been formulated from a geometric point of view, and the further subdivision of Lie–remarkable equations into strong and weak Lie–remarkable equations has been considered.

2 Second order Monge-Ampère equations

The second order Monge-Ampère equation in two independent variables, introduced by Ampère[16] in 1815, has the form

$$Hu_{tt} + 2Ku_{tx} + Lu_{xx} + M + N(u_{tt}u_{xx} - u_{tx}^2) = 0, \tag{9}$$

where the coefficients H, K, L, M, N ($N \neq 0$) depend on t, x, u, u_t, u_x.

In 1968, Boillat[17] discovered that (9) is the only second order equation possessing the property of complete exceptionality in the Lax[18] sense.

The property of complete exceptionality has been used to derive Monge-Ampère equations involving more than two independent variables (see Refs. [19, 20, 21]). Given an unknown field $u(x_0, x_1, \dots, x_n)$ (x_0 denoting the time), and its associated Hessian matrix, the most general second order PDE being completely exceptional (and called Monge-Ampère equation) is provided by a linear combination of all minors extracted from the Hessian matrix, with coefficients depending at most on x_α, u and first order derivatives of u.

Here we want to stress that Monge-Ampère equations, in addition to the complete exceptionality, possess another remarkable property, that of being uniquely characterized by their Lie point symmetries. We first analyze the classical Monge-Ampère equation in two independent variables, then we will consider generalizations to three independent variables and to higher order equations.

3 Theorem. *The classical Monge-Ampère equation written in the form*

$$\kappa_1(u_{tt}u_{xx} - u_{tx}^2) + \kappa_2 u_{tt} + \kappa_3 u_{tx} + \kappa_4 u_{xx} + \kappa_5 = 0, \tag{10}$$

where the coefficients κ_i ($\kappa_1 \neq 0$) are constant, is Lie–remarkable.

Proof. In fact, Eq. (10) is invariant with respect to a Lie group of point transformations spanning a 9–dimensional Lie Algebra $\mathcal{L}$ generated by the following vector fields:

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial t}, \qquad \Xi_2 = \frac{\partial}{\partial x}, \qquad \Xi_3 = \frac{\partial}{\partial u},\\
&\Xi_4 = t\frac{\partial}{\partial u}, \qquad \Xi_5 = x\frac{\partial}{\partial u},
\end{aligned}$$

$$\begin{aligned}
\Xi_6 &= \kappa_1 t\frac{\partial}{\partial t} - \kappa_1 x\frac{\partial}{\partial x} + (\kappa_2 x^2 - \kappa_4 t^2)\frac{\partial}{\partial u},\\
\Xi_7 &= 2\kappa_1 t\frac{\partial}{\partial x} + (\kappa_3 t^2 - 2\kappa_2 xt)\frac{\partial}{\partial u},\\
\Xi_8 &= 2\kappa_1 x\frac{\partial}{\partial x} + (-\kappa_2 x^2 + \kappa_4 t^2 + 2\kappa_1 u)\frac{\partial}{\partial u},\\
\Xi_9 &= 2\kappa_1 x\frac{\partial}{\partial t} + (\kappa_3 x^2 - 2\kappa_4 xt)\frac{\partial}{\partial u}.
\end{aligned}$$

If we want to look for the second order partial differential equation

$$\Delta(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0, \tag{11}$$

where Δ is an unspecified function of the indicated arguments, admitting the Lie algebra $\mathcal{L}$, we need to find the associated second order differential invariants. From a geometric point of view, a scalar second order partial differential equation in two independent variables characterizes a submanifold of dimension 7 in the 8–dimensional jet space whose coordinates are $(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx})$.

The second order prolongations of the admitted vector fields give rise to a distribution of rank 7, and this implies that a single second order differential invariant exists, say

$$I = \kappa_1(u_{tt}u_{xx} - u_{tx}^2) + \kappa_2 u_{tt} + \kappa_3 u_{tx} + \kappa_4 u_{xx}. \tag{12}$$

Therefore, the most general second order partial differential equation left invariant by the Lie algebra of point transformations of Eq. (10) has the form

$$\Delta(I) = 0, \tag{13}$$

whereupon it follows

$$I = \kappa = \text{constant}; \tag{14}$$

thus, we have an equation belonging to the same class as (10); by choosing $\kappa = -\kappa_5$, we recover exactly the Monge-Ampère equation (10). □

Let us now consider the second order Monge-Ampère equation in three independent variables[19]

$$\begin{aligned}
&\kappa_1[u_{tt}(u_{xx}u_{yy} - u_{xy}^2) + u_{tx}(u_{ty}u_{xy} - u_{tx}u_{yy} + u_{ty}(u_{tx}u_{xy} - u_{ty}u_{xx})]\\
&+\kappa_2(u_{xx}u_{yy} - u_{xy}^2) + \kappa_3(u_{ty}u_{xy} - u_{tx}u_{yy}) + \kappa_4(u_{tx}u_{xy} - u_{ty}u_{xx})\\
&+\kappa_5(u_{tt}u_{yy} - u_{ty}^2) + \kappa_6(u_{tx}u_{ty} - u_{tt}u_{xy}) + \kappa_7(u_{tt}u_{xx} - u_{tx}^2)\\
&+\kappa_8 u_{tt} + \kappa_9 u_{tx} + \kappa_{10} u_{ty} + \kappa_{11} u_{xx} + \kappa_{12} u_{xy} + \kappa_{13} u_{yy} + \kappa_{14} = 0,
\end{aligned} \tag{15}$$

where κ_i $(i = 1, \dots, 14)$ are taken constant.

The explicit determination of the infinitesimal generators of the admitted Lie group results quite complicated and the use of Computer Algebra packages[7] reveals extremely memory consuming since the expression of the infinitesimals involves thousands of terms.

Without loss of generality, it is possible to introduce the substitution

$$u \to u + \alpha_1 t^2 + \alpha_2 tx + \alpha_3 ty + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2,$$

where α_i are suitable constants, and reduce Eq. (15) to an equivalent form where the linear terms in the derivatives disappear, that is, we may consider Eq. (15) with $\kappa_8 = \kappa_9 = \kappa_{10} = \kappa_{11} = \kappa_{12} = \kappa_{13} = 0$. Since the introduced transformation is invertible, both equations admit the same point symmetries.

In general, the Lie algebra of point symmetries of Eq. (15) (with $\kappa_8 = \kappa_9 = \kappa_{10} = \kappa_{11} = \kappa_{12} = \kappa_{13} = 0$) is 11–dimensional; since Eq. (15) represents a 12–dimensional submanifold in the 13–dimensional jet space with coordinates $(t, x, y, u, u_t, u_x, u_y, u_{tt}, u_{tx}, u_{ty}, u_{xx}, u_{xy}, u_{yy})$, a 12–dimensional distribution generated by point symmetries is needed in order to determine a single second order differential invariant. Consequently, Eq. (15) can not be in general Lie–remarkable. Nevertheless, the following theorem can be proved.

4 Theorem. *Equation (15) (with $\kappa_8 = \kappa_9 = \kappa_{10} = \kappa_{11} = \kappa_{12} = \kappa_{13} = 0$), when the coefficients are such that*

$$\kappa_1 = 1, \qquad \kappa_2\kappa_6^2 - \kappa_3\kappa_4\kappa_6 + \kappa_4^2\kappa_5 - (4\kappa_2\kappa_5 - \kappa_3^2)\kappa_7 = 0, \tag{16}$$

is Lie–remarkable.

Proof. When the conditions (16) are satisfied, the Lie algebra admitted by Eq. (15) is 13–dimensional and is spanned by the following vector fields:

$$\begin{aligned}
\Xi_1 &= \frac{\partial}{\partial t} \qquad \Xi_2 = \frac{\partial}{\partial x}, \qquad \Xi_3 = \frac{\partial}{\partial y}, \qquad \Xi_4 = \frac{\partial}{\partial u},\\
\Xi_5 &= t\frac{\partial}{\partial u}, \qquad \Xi_6 = x\frac{\partial}{\partial u}, \qquad \Xi_7 = y\frac{\partial}{\partial u},\\
\Xi_8 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2u\frac{\partial}{\partial u},\\
\Xi_9 &= \left(x + \frac{\kappa_3\kappa_4 - 2\kappa_2\kappa_6}{\kappa_3^2 - 4\kappa_2\kappa_5}y\right)\frac{\partial}{\partial t}\\
&\quad - \frac{(2\kappa_2 t + \kappa_3 x + \kappa_4 y)(\kappa_3^2 x + \kappa_3\kappa_4 y - 2\kappa_2(2\kappa_5 x + \kappa_6 y))}{2(\kappa_3^2 - 4\kappa_2\kappa_5)}\frac{\partial}{\partial u},\\
\Xi_{10} &= (2\kappa_4\kappa_5 - \kappa_3\kappa_6)y\frac{\partial}{\partial t} + (2\kappa_3^2 x + \kappa_3\kappa_4 y - 2\kappa_2(4\kappa_5 x + \kappa_6 y))\frac{\partial}{\partial x}\\
&\quad + ((\kappa_3^2 - 4\kappa_2\kappa_5)y\frac{\partial}{\partial y} + (\kappa_3^2 - 4\kappa_2\kappa_5)(2u + \kappa_2 t^2 - \kappa_5 x^2)\\
&\quad - 2(\kappa_3\kappa_4\kappa_5 x + \kappa_2(2\kappa_4\kappa_5 t - \kappa_6(\kappa_3 t + 2\kappa_5 x)))y\\
&\quad - (\kappa_4^2\kappa_5 - \kappa_2\kappa_6^2)y^2)\frac{\partial}{\partial u},\\
\Xi_{11} &= -2(\kappa_3\kappa_4 - 2\kappa_2\kappa_6)(-2\kappa_4\kappa_5 + \kappa_3\kappa_6)y\frac{\partial}{\partial t}\\
&\quad + 2(\kappa_3^2 - 4\kappa_2\kappa_5)(\kappa_3\kappa_4 - 2\kappa_2\kappa_6)x\frac{\partial}{\partial x}\\
&\quad + 2(\kappa_3^2 - 4\kappa_2\kappa_5)(\kappa_3^2 x + 2\kappa_3\kappa_4 y - 4\kappa_2(\kappa_5 x + \kappa_6 y))\frac{\partial}{\partial y}\\
&\quad + (-(\kappa_3^2 - 4\kappa_2\kappa_5)(-2(\kappa_3\kappa_4 - 2\kappa_2\kappa_6)(2u + \kappa_2 t^2)\\
&\quad + 2\kappa_2(-2\kappa_4\kappa_5 + \kappa_3\kappa_6)tx + (\kappa_3^2 - 4\kappa_2\kappa_5)\kappa_6 x^2)\\
&\quad - 2(\kappa_2(-\kappa_3\kappa_4 + 2\kappa_2\kappa_6)(-2\kappa_4\kappa_5 + \kappa_3\kappa_6)t
\end{aligned}$$

$$
\begin{aligned}
&+(4\kappa_2\kappa_4^2\kappa_5^2+\kappa_3\kappa_4(\kappa_3^2-8\kappa_2\kappa_5)\kappa_6\\
&+\kappa_2(-\kappa_3^2+8\kappa_2\kappa_5)\kappa_6^2)x)y-\kappa_6(\kappa_3\kappa_4-2\kappa_2\kappa_6)^2y^2)\frac{\partial}{\partial u},\\
\Xi_{12}&=-2(-2\kappa_4\kappa_5+\kappa_3\kappa_6)^2y\frac{\partial}{\partial t}+2(\kappa_3^2-4\kappa_2\kappa_5)(-2\kappa_4\kappa_5+\kappa_3\kappa_6)x\frac{\partial}{\partial x}\\
&+2(\kappa_3^2-4\kappa_2\kappa_5)(\kappa_3^2t+2\kappa_3\kappa_6y-4\kappa_5(\kappa_2t+\kappa_4y))\frac{\partial}{\partial y}\\
&-(\kappa_3^2-4\kappa_2\kappa_5)(-2\kappa_6(2\kappa_3u+\kappa_2\kappa_3t^2+2\kappa_2\kappa_5tx)\\
&+\kappa_4(8\kappa_5u+\kappa_3^2t^2+2\kappa_3\kappa_5tx))\\
&-2(\kappa_3\kappa_4-2\kappa_2\kappa_6)(\kappa_3^2\kappa_6t-2\kappa_5(\kappa_2\kappa_6t+\kappa_4\kappa_5x)\\
&+\kappa_3(-\kappa_4\kappa_5t+\kappa_5\kappa_6x))y\\
&+\kappa_6(-\kappa_3\kappa_4+2\kappa_2\kappa_6)(-2\kappa_4\kappa_5+\kappa_3\kappa_6)y^2\frac{\partial}{\partial u},\\
\Xi_{13}&=2(-2\kappa_4\kappa_5+\kappa_3\kappa_6)^2y\frac{\partial}{\partial t}\\
&2(\kappa_3^2-4\kappa_2\kappa_5)(\kappa_3\kappa_4t-2\kappa_2\kappa_6t+4\kappa_4\kappa_5x-2\kappa_3\kappa_6x)\frac{\partial}{\partial x}\\
&-2(\kappa_3^2-4\kappa_2\kappa_5)(-2\kappa_4\kappa_5y+\kappa_2\kappa_6y)\frac{\partial}{\partial y}\\
&-(\kappa_3^2-4\kappa_2\kappa_5)(2\kappa_6(2\kappa_3u-2\kappa_2\kappa_5tx-\kappa_3\kappa_5x^2)\\
&+\kappa_4(\kappa_3^2t^2+2\kappa_5(-4u-2\kappa_2t^2+\kappa_3tx+2\kappa_5x^2)))\\
&-2((\kappa_4^2\kappa_5(-\kappa_3^2+8\kappa_2\kappa_5)+\kappa_3\kappa_4(\kappa_3^2-8\kappa_2\kappa_5)\kappa_6+4\kappa_2^2\kappa_5\kappa_6^2)t\\
&+\kappa_5(\kappa_3\kappa_4-2\kappa_2\kappa_6)(2\kappa_4\kappa_5-\kappa_3\kappa_6)x)y\\
&-\kappa_4(-2\kappa_4\kappa_5+\kappa_3\kappa_6)^2y^2\frac{\partial}{\partial u}.
\end{aligned}
$$

The second order prolongations of these vector fields give rise to a distribution of rank 12, and we obtain the following second order differential invariant:

$$
\begin{aligned}
I &= (u_{tt}u_{xx}u_{yy}-u_{tt}u_{xy}^2-u_{tx}^2u_{yy}+2u_{tx}u_{ty}u_{xy}-u_{ty}^2u_{xx})\\
&+ \kappa_2(u_{xx}u_{yy}-u_{xy}^2)-\kappa_3(u_{tx}u_{yy}-u_{ty}u_{xy})+\kappa_4(u_{tx}u_{xy}-u_{ty}u_{xx})\\
&+ \kappa_5(u_{tt}u_{yy}-u_{ty}^2)-\kappa_6(u_{tt}u_{xy}-u_{tx}u_{ty})+\kappa_7(u_{tt}u_{xx}-u_{tx}^2).
\end{aligned}
\tag{17}
$$

Therefore, the most general second order partial differential equation in three independent variables left invariant with respect to the Lie symmetries of Eq. (15), along with the conditions (16), has the form

$$\Delta(I)=0, \tag{18}$$

where Δ is an arbitrary function of its argument. Similar reasonings as before allow us to say that Eq. (15), when the conditions (16) hold, is Lie–remarkable. □

More generally, it is possible to prove that second order Monge-Ampère equations involving more than three independent variables, when some conditions involving their coefficients are satisfied, are instances of Lie–remarkable equations.

3 Higher order Monge-Ampère equations

The property of complete exceptionality has been used by Boillat[22] to determine higher order Monge-Ampère equations for the unknown $u(t, x)$.

By considering an equation of order $N > 2$, it is necessary to distinguish the case where N is even from the case where N is odd. If $N = 2M$, the most general nonlinear completely exceptional equation is given by a linear combination of all minors, including the determinant, of the following Hankel matrix:

$$H = \begin{bmatrix} X_0 & X_1 & X_2 & \dots & X_{M-1} & X_M \\ X_1 & X_2 & X_3 & \dots & X_M & X_{M+1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ X_{M-1} & X_M & X_{M+1} & \dots & X_{2M-2} & X_{2M-1} \\ X_M & X_{M+1} & X_{M+2} & \dots & X_{2M-1} & X_{2M} \end{bmatrix}, \tag{19}$$

where $X_i = \dfrac{\partial^N u}{\partial t^i \partial x^{N-i}}$. In the case where $N = 2M - 1$, we have to consider the linear combination of all minors extracted from (19) where the last row has been removed. In both cases the coefficients of the linear combination are functions of t, x, u and its derivatives up to the order $N - 1$. In the following we shall limit ourselves to the case where these coefficients are constant. By considering the third order Monge-Ampère equation

$$\begin{aligned} &\widetilde{\kappa}_1(u_{ttx}u_{xxx} - u_{txx}^2) + \widetilde{\kappa}_2(u_{ttt}u_{xxx} - u_{ttx}u_{txx}) + \widetilde{\kappa}_3(u_{ttt}u_{txx} - u_{ttx}^2) \\ &+\widetilde{\kappa}_4 u_{ttt} + \widetilde{\kappa}_5 u_{ttx} + \widetilde{\kappa}_6 u_{txx} + \widetilde{\kappa}_7 u_{xxx} + \widetilde{\kappa}_8 = 0, \end{aligned} \tag{20}$$

the substitution

$$u \to u + \alpha_1 t^3 + \alpha_2 t^2 x + \alpha_3 t x^2 + \alpha_4 x^3$$

provides the equation

$$\kappa_1(u_{ttx}u_{xxx} - u_{txx}^2) + \kappa_2(u_{ttt}u_{xxx} - u_{ttx}u_{txx}) + \kappa_3(u_{ttt}u_{txx} - u_{ttx}^2) = \kappa. \tag{21}$$

The Lie algebra of point symmetries of Eq. (21) is 10–dimensional; since this equation is a 11–dimensional submanifold in the 12–dimensional jet space with coordinates $(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, u_{ttt}, u_{ttx}, u_{txx}, u_{xxx})$, a 11–dimensional distribution generated by point symmetries is needed in order to determine a single second order differential invariant. Therefore, Eq. (21) is not in general Lie–remarkable. Nevertheless, the following theorem can be proved.

5 Theorem. *The equation*

$$(u_{ttx}u_{xxx} - u_{txx}^2) + \lambda(u_{ttt}u_{xxx} - u_{ttx}u_{txx}) + \lambda^2(u_{ttt}u_{txx} - u_{ttx}^2) = \mu, \tag{22}$$

where

$$\lambda = \frac{\kappa_3}{\kappa_2}, \qquad \mu = \frac{\kappa\kappa_3}{\kappa_2^2},$$

obtained from (21) by choosing $\kappa_1 = \dfrac{\kappa_2^2}{\kappa_3}$, *is Lie–remarkable.*

Proof. The Lie algebra of point symmetries admitted by (22) is infinite–dimensional and is spanned by the vector fields

$$
\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial t}, \qquad \Xi_2 = \frac{\partial}{\partial x}, \qquad \Xi_3 = \frac{\partial}{\partial u},\\
&\Xi_4 = (2t - 3\lambda x)\frac{\partial}{\partial t} - x\frac{\partial}{\partial x}, \qquad \Xi_5 = \lambda^2 x\frac{\partial}{\partial t} + (2t - \lambda x)\frac{\partial}{\partial x},\\
&\Xi_6 = \lambda x\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}, \qquad \Xi_7 = t\frac{\partial}{\partial u}, \qquad \Xi_8 = x\frac{\partial}{\partial u},\\
&\Xi_9 = t^2\frac{\partial}{\partial u}, \qquad \Xi_{10} = tx\frac{\partial}{\partial u}, \qquad \Xi_{11} = x^2\frac{\partial}{\partial u},\\
&\Xi_{12} = F(t - \lambda x)\frac{\partial}{\partial u},
\end{aligned}
\tag{23}
$$

where F is an arbitrary function of $(t - \lambda x)$.

The third order prolongations of these operators give rise to a distribution of rank 11; therefore, we are able to determine the unique third order differential invariant

$$
I = (u_{ttx}u_{xxx} - u_{txx}^2) + \lambda(u_{ttt}u_{xxx} - u_{ttx}u_{txx}) + \lambda^2(u_{ttt}u_{txx} - u_{ttx}^2), \tag{24}
$$

and this completes the proof. □

We may also consider fourth order Monge-Ampère equations and state, for instance, the following theorem.

6 Theorem. *The fourth order Monge-Ampère equation*

$$
u_{tttt}(u_{ttxx}u_{xxxx} - u_{txxx}^2) + 2u_{tttx}u_{ttxx}u_{txxx} - u_{ttxx}^3 - u_{tttx}^2 u_{xxxx} = \kappa, \tag{25}
$$

where κ is constant, is Lie–remarkable.

Proof. In fact, the Lie algebra of point symmetries admitted by Eq. (25) is 16–dimensional and is spanned by the vector fields

$$
\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial t}, \qquad \Xi_2 = \frac{\partial}{\partial x}, \qquad \Xi_3 = \frac{\partial}{\partial u},\\
&\Xi_4 = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x}, \qquad \Xi_5 = x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}, \qquad \Xi_6 = x\frac{\partial}{\partial t},\\
&\Xi_7 = t\frac{\partial}{\partial x}, \qquad \Xi_8 = t\frac{\partial}{\partial u}, \qquad \Xi_9 = x\frac{\partial}{\partial u},\\
&\Xi_{10} = t^2\frac{\partial}{\partial u}, \qquad \Xi_{11} = tx\frac{\partial}{\partial u}, \qquad \Xi_{12} = x^2\frac{\partial}{\partial u},\\
&\Xi_{13} = t^3\frac{\partial}{\partial u}, \qquad \Xi_{14} = t^2 x\frac{\partial}{\partial u}, \qquad \Xi_{15} = tx^2\frac{\partial}{\partial u}, \qquad \Xi_{16} = x^3\frac{\partial}{\partial u}.
\end{aligned}
\tag{26}
$$

Their fourth order prolongations give rise to a distribution of rank 16, whereupon we have uniquely the fourth order differential invariant

$$
I = u_{tttt}(u_{ttxx}u_{xxxx} - u_{txxx}^2) + 2u_{tttx}u_{ttxx}u_{txxx} - u_{ttxx}^3 - u_{tttx}^2 u_{xxxx}, \tag{27}
$$

and this enables us to say that Eq. (25) is Lie–remarkable. □

By similar arguments, it is possible to prove that the property of being uniquely characterized by their Lie point symmetries is also shared by some Monge–Ampère equations of order higher than the fourth.

4 Minimal surface equation in $\mathbb{R}^3$

The study of minimal surface equations in $\mathbb{R}^3$ dates back to Lagrange (1762) who posed the problem of determining a graph over an open set $\Omega \subseteq \mathbb{R}^2$ with the least possible area among all surfaces assuming given values on $\partial\Omega$. Meusnier (1776) gave a geometric interpretation of the minimal graph equation recognizing that their mean curvature vanishes. Also, in the middle of 19-th century Plateau observed that minimal surfaces can be physically realized as soap films.

In $\mathbb{R}^3$, at least locally, a minimal surface can be represented in the form $z = u(x, y)$, where the function u satisfies a quasilinear elliptic second order partial differential equation:

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0, \qquad \nabla = (\partial_x, \partial_y)\,,$$

i.e.,

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0, \tag{28}$$

that, through the substitution $y \to it$, provides the hyperbolic 2D Born-Infeld equation

$$(1 - u_t^2)u_{xx} + 2u_x u_t u_{xt} - (1 + u_x^2)u_{tt} = 0. \tag{29}$$

Minimal surface equation is invariant with respect to a Lie group of point transformations whose vector fields

$$\begin{aligned} &\Xi_1 = \frac{\partial}{\partial x}, \qquad \Xi_2 = \frac{\partial}{\partial x}, \qquad \Xi_3 = \frac{\partial}{\partial u}, \\ &\Xi_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}, \qquad \Xi_5 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \\ &\Xi_6 = u\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}, \qquad \Xi_7 = u\frac{\partial}{\partial y} - y\frac{\partial}{\partial u} \end{aligned} \tag{30}$$

span a 7–dimensional Lie Algebra.

Since the second order prolongations of (30) give rise to a distribution of rank 7, we have only one second order differential invariant[2], say:

$$I = \frac{\left((1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy}^2\right)^2}{(1 + u_x^2 + u_y^2)(u_{xx}u_{yy} - u_{xy}^2)}. \tag{31}$$

The most general equation which is invariant with respect to the given Lie group must be a function of this invariant. If we want to have an equation in normal form we have to set

$$I = 0,$$

so recovering exactly the minimal surface equation.

[2]R. Tracinà, private communication.

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