

Variational Multivectors and Brackets in the Geometry of Jet Spaces

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Abstract

This is a summary of our recent results to appear in [5]. In the framework of jet superspace geometry, we introduce variational multivectors and the variational Poisson and Schouten brackets, which are dual to the variational complex. Their relations with the antibracket in field theory and applications to finding Hamiltonian structures for evolution equations are outlined. We show also that local variational differential operators of B. L. Voronov, I. V. Tyutin, and Sh. S. Shakhverdiev are determined by variational multivectors and are well defined on the algebra of multilocal functionals. To achieve the latter result, we show that locally a finite number of smooth action functionals are either linearly dependent or ‘almost’ functionally independent in a certain sense.

Introduction

In the geometric theory of jet spaces the so-called *variational complex* [12, 13] is of central importance (see, for example, [2, 7] for an introductory treatment). The variational complex is obtained from the de Rham sequence on jet manifolds by

a quotient with respect to ‘variationally trivial’ forms. This complex contains the Euler operator as one of its differentials.

By analogy with ordinary manifolds, where symmetric and skew-symmetric products of vector fields and their Poisson and Schouten brackets are introduced as dual counterparts of forms and the de Rham differential, it is interesting to consider a dual theory to the variational complex and its differential: *variational multivectors* and their brackets, the *variational Poisson and Schouten brackets*.

In this note, we summarize our recent results on this topic; full details can be found in [5].

In Section 1, we set up notation and terminology related to graded vector spaces, graded algebras, and differential operators over such algebras.

Then, we introduce a graded Lie algebra structure on a space of multilinear maps on a vector space. This structure generalizes the Nijenhuis-Richardson and Schouten brackets. By interpreting multilinear maps as differential operators on polynomial functions, we provide an algebraic counterpart for a class of operators recently introduced in [14], namely the *local variational differential operators*.

In Section 2, we specialize our algebraic setting to a geometric situation. Namely, we consider differential operators on *local functionals* (or *actions*). Local functionals are elements of a cohomology group of the *horizontal de Rham complex*. This complex consists of differential forms on the base manifold with coefficients in functions of dependent variables (fields) and a finite number of their derivatives. In other words, local functionals are Lagrangian densities modulo total divergences. The distinguished expressions of variational multivectors and their bracket are given in Theorems 2, 3.

In Section 3, we consider the above general formalism in different contexts.

First, we observe that the variational Poisson bracket coincides with the Poisson brackets constructed by Kupershmidt [8] on the so-called ‘cotangent bundle to a bundle’ when restricted to the subspace of polynomial functions.

Second, we consider the skew-symmetric variational multivectors and the *variational Schouten bracket*. The coordinate version of this bracket is well known under the name *antibracket* (see, e.g., [4] and references therein), so that we obtain here a geometric setting for the antibracket formalism.

This bracket is also of great importance in the Hamiltonian theory of integrable evolution equations, since skew-symmetric variational bivectors whose variational Schouten bracket with itself vanishes are just Hamiltonian operators (see, e.g., [3] and references therein).

As an application of our approach to variational multivectors we discuss the *calculus of local variational differential operators* by B. L. Voronov, I. V. Tyutin, and Sh. S. Shakhverdiev [14]. In the study of the so-called $\delta(0)$ -problem in field theory, the authors discovered fundamentally new operators, which act on smooth functions $F(S_1, \dots, S_N)$ of local functionals S_i (*multilocal functionals*). We prove that these operators are well defined, i.e., if $F(S_1, \dots, S_N)$ is an identically zero functional then $\nabla(F(S_1, \dots, S_N))$ is also zero. The crucial part of the proof is to

show that all relations in the algebra of multilocal functionals are linear, roughly speaking. This fact looks interesting by itself.

1 The Lie algebra of multilinear mappings: an algebraic model

Graded spaces and graded algebras. Let G be an Abelian group. A vector space V is called G -graded if $V = \bigoplus_{g \in G} V_g$ for some vector spaces V_g . *Superspaces* are \mathbb{Z}_2 -graded vector spaces. The notion of graded space can be carried out in a natural way to subspaces, tensor products, etc. of graded spaces.

A \mathbb{k} -algebra A is called G -graded if A is a G -graded vector space and $A_{g_1}A_{g_2} \subset A_{g_1+g_2}$, for all $g_1, g_2 \in G$.

A *commutation factor* is a pairing $G \times G \rightarrow \mathbb{k} \setminus \{0\}$, $(g_1, g_2) \mapsto \{g_1, g_2\}$, such that

$$\{g_1, g_2\}^{-1} = \{g_2, g_1\}, \quad \{g_1 + g_2, g_3\} = \{g_1, g_3\}\{g_2, g_3\}.$$

When $G = \mathbb{Z}$ or \mathbb{Z}_2 , there is only one nontrivial commutation factor, namely, the *super-commutation factor* $\{g_1, g_2\} = (-1)^{g_1 g_2}$. In formulas commutation factors are used according to the following ‘generalized rule of signs’: whenever an object (i.e., an element of a G -graded vector space) of degree g_1 is interchanged with an object of degree g_2 , the multiplier $\{g_1, g_2\}$ is introduced.

For a fixed element $g = (g_1, \dots, g_n) \in G^n = G \oplus \dots \oplus G$ there is a unique function $\epsilon_g: S_n \rightarrow \mathbb{k} \setminus \{0\}$ on the permutation group S_n such that

1. $\epsilon_g(\sigma_i) = \{g_i, g_{i+1}\}$ for a transposition $\sigma_i = (i, i+1)$;
2. $\epsilon_g(\sigma' \circ \sigma'') = \epsilon_{\sigma''(g)}(\sigma')\epsilon_g(\sigma'')$, where $\sigma(g) = (g_{\sigma(1)}, \dots, g_{\sigma(n)})$.

For the super-commutation factor the function $\epsilon_{(1, \dots, 1)}$ coincides with the standard sign of permutations. We shall denote it by ϵ .

For our convenience, if $v \in V_{g_1}$ and $w \in W_{g_2}$ are two homogeneous elements of two G -graded vector spaces V and W , then we write $\{v, w\}$ rather than $\{g_1, g_2\}$.

A G -graded algebra A is called *commutative* if for all $a, b \in A$ we have $ab = \{a, b\}ba$.

A left module M over a G -graded commutative algebra A is called G -graded if $M = \bigoplus_{g \in G} M_g$ and $A_{g_1}M_{g_2} \subset M_{g_1+g_2}$.

A G -graded *Lie algebra* is a G -graded algebra A such that the multiplication in A , denoted by $[\cdot, \cdot]: A \otimes_{\mathbb{k}} A \rightarrow A$, satisfies the properties:

$$[a, b] = -\{a, b\}[b, a], \quad [[a, b], c] = [a, [b, c]] + \{b, c\}[[a, c], b],$$

for all $a, b, c \in A$. For example, if V is a G -graded vector space, then $\text{Hom}_{\mathbb{k}}(V, V)$, equipped with the commutator, is a G -graded Lie algebra.

A *Lie superalgebra* is \mathbb{Z}_2 -graded Lie algebra with respect to the super-commutation factor.

Let A be an associative commutative G -graded algebra with unity. A \mathbb{k} -homomorphism $\Delta \in \text{Hom}_{\mathbb{k}}(A, A)$ is called a *scalar G -graded differential operator of order k* , if for all $a_0, \dots, a_k \in A$ we have

$$[a_0, [a_1, \dots, [a_k, \Delta] \dots]] = 0.$$

In this equality a_i are the operators of left multiplication. Denote by $\text{Diff}_k(A)$ the set of all scalar differential operators of order k . It is clear that $\text{Diff}_0(A) = A$ and $\text{Diff}_k(A) \subset \text{Diff}_l(A)$ for $k \leq l$. If $\Delta_1 \in \text{Diff}_k(A)$ and $\Delta_2 \in \text{Diff}_l(A)$, then it is easily seen that $\Delta_1 \circ \Delta_2 \in \text{Diff}_{k+l}(A)$ and $[\Delta_1, \Delta_2] \in \text{Diff}_{k+l-1}(A)$.

The Lie algebra of multilinear mappings. Let V be a G -graded vector space. Let $\{\mathcal{M}_k(V)\}_{k \in \mathbb{N}}$ be a family of graded subspaces of the space of k -linear maps of V with values in V with the following properties:

1. $\mathcal{M}_0(V) = V$;
2. for all $f \in \mathcal{M}_k(V)$, $g \in \mathcal{M}_l(V)$, and $1 \leq i \leq k$ the $(k+l-1)$ -linear map h defined by

$$h(v_1, \dots, v_{k+l-1}) = f(v_1, \dots, v_{i-1}, g(v_i, \dots, v_{i+l-1}), v_{i+l}, \dots, v_{k+l-1})$$

belongs to $\mathcal{M}_{k+l-1}(V)$.

The space $\mathcal{M}_1(V)$ is a graded Lie algebra with respect to the commutator. Let \mathfrak{g} be a graded Lie subalgebra of $\mathcal{M}_1(V)$. For each $k \geq 1$ define G -graded vector space $\mathfrak{g}^{(k)}$ to be the graded subspace of $\mathcal{M}_k(V)$ such that $f \in \mathfrak{g}^{(k)}$ if

1. f is (graded) symmetric, i.e., $f(v_1, \dots, v_k) = \epsilon_v(\sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)})$ for all $\sigma \in S_k$;
2. for all v_1, \dots, v_{k-1} the maps $v \mapsto f(v_1, \dots, v_{i-1}, v, v_i, \dots, v_{k-1})$ belong to \mathfrak{g} .

Obviously, $\mathfrak{g}^{(1)} = \mathfrak{g}$. By definition, we put $\mathfrak{g}^{(0)} = V$ and $\mathfrak{g}^{(k)} = 0$ for $k < 0$.

Let us introduce the following notation: if f is a k -linear map as above and $v_i \in V$, then $f(v_1, \dots, v_l)$ for $l \leq k$ will stand for the $(k-l)$ -linear map obtained by contracting the first l arguments of f with v_i . Then, we can prove the following theorem [5].

Theorem 1. *On the space $\mathfrak{g}^{(*)} = \bigoplus_k \mathfrak{g}^{(k)}$ there exists a unique G -graded Lie algebra structure $[[\cdot, \cdot]]$ such that*

- 1 $[[\mathfrak{g}^{(k)}, \mathfrak{g}^{(l)}]] \subset \mathfrak{g}^{(k+l-1)}$;
- 2 $[[f, v]] = f(v)$ for $v \in \mathfrak{g}^{(0)} = V$, $f \in \mathfrak{g}^{(k)}$;

Using the induction, we get the following explicit formula for this bracket:

$$\begin{aligned} \llbracket f_1, f_2 \rrbracket(v_1, \dots, v_{k+l-1}) &= \sum_{\sigma \in S_{k+l-1}^l} \epsilon_v(\sigma) f_1(f_2(v_{\sigma(1)}, \dots, v_{\sigma(l)}, v_{\sigma(l+1)}, \dots, v_{\sigma(k+l-1)})) \\ &\quad - \{f_1, f_2\} \sum_{\sigma \in S_{k+l-1}^k} \epsilon_v(\sigma) f_2(f_1(v_{\sigma(1)}, \dots, v_{\sigma(k)}, v_{\sigma(k+1)}, \dots, v_{\sigma(k+l-1)})), \end{aligned}$$

where $S_n^i \subset S_n$ is the set of all $(i, n-i)$ -*unshuffles*¹, that is, all permutations $\sigma \in S_n$ such that $\sigma(1) < \sigma(2) < \dots < \sigma(i)$ and $\sigma(i+1) < \sigma(i+2) < \dots < \sigma(n)$.

It can be recognized that, by choosing suitable graded vector spaces and algebras, one recovers the *Nijenhuis-Richardson bracket*, the *Schouten bracket* and the *symmetric Schouten concomitant* (see [5] for details and references).

Consider the ideal I in the tensor algebra $T(V)$ generated by $(v \otimes w - \{v, w\}w \otimes v) \in V^{\otimes 2}$ for $v, w \in V$. The algebra $S(V) = T(V)/I$ is a G -graded associative commutative algebra with unity called the *symmetric algebra* of V .

Now, to multilinear maps $f \in \mathfrak{g}^{(k)}$ we assign differential operators on the algebra $S(V)$. Namely, for $k = 0$ we define ∇_f to be the multiplication by $f \in V$. For $k > 0$ it can be proved [5] that there exists a unique differential operator $\nabla_f \in \text{Diff}_k(S(V))$ such that $\nabla_f|_{S^l(V)} = 0$, for $0 \leq l < k$, and $\nabla_f(v_1 \cdots v_k) = f(v_1, \dots, v_k)$. The operator ∇_f fulfills the expression

$$\nabla_f(v_1 \cdots v_n) = \sum_{\sigma \in S_n^k} \epsilon_v(\sigma) f(v_{\sigma(1)} \cdots v_{\sigma(k)}) v_{\sigma(k+1)} \cdots v_{\sigma(n)}$$

for $n > k$. As one can expect, we have $[\nabla_{f_1}, \nabla_{f_2}] = \nabla_{\llbracket f_1, f_2 \rrbracket}$ for $f_1 \in \mathfrak{g}^{(k)}$ and $f_2 \in \mathfrak{g}^{(l)}$.

Now we extend the operators ∇_f to a bigger algebra. For this purpose, let us represent the graded space V in the form $V = V_0 \oplus V_+$, where $V_+ = \bigoplus_{g \in G \setminus \{0\}} V_g$. We have $S(V) = S(V_0) \otimes_{\mathbb{k}} S(V_+)$. The algebra $S(V_0)$ can be identified with an algebra of (polynomial) functions on the dual space V_0^* . Let us extend this algebra to the algebra \mathcal{A}_0 of functions on V_0^* of the form $F(v_1, \dots, v_N)$, where $F \in C^\infty(\mathbb{R}^N)$ is a smooth function in many arguments and $v_i \in V_0$. Assign to all elements of \mathcal{A}_0 the degree 0. Denote by $\mathcal{A}(V)$ the graded algebra $\mathcal{A}_0 \otimes_{\mathbb{k}} S(V_+)$. Then, it is not difficult to prove that, for each $f \in \mathfrak{g}^{(k)}$, the operator ∇_f has the unique extension

$$\nabla_f(F(v_1, \dots, v_N) \otimes v_+) = \sum_{\substack{0 \leq l \leq k \\ \sigma \in S_N^l}} \frac{\partial^l F}{\partial t_{\sigma(1)} \cdots \partial t_{\sigma(l)}}(v_1, \dots, v_N) \otimes \nabla_{f(v_{\sigma(1)}, \dots, v_{\sigma(l)})}(v_+),$$

to the algebra $\mathcal{A}(V)$.

¹The term *unshuffle* is borrowed from [9] and means separating an ordered set into two subsets, the order within each subset being as in the original set.

2 Variational Poisson bracket

The jet bundle setting: preliminaries. In this paper we deal with the infinite jet bundles of vector *superbundles*. Jets of purely even bundles have been detailed extensively in the literature (see, e.g., [2, 7]), so we will shortly describe the graded setting here. We assume that all manifolds and maps are C^∞ .

We say that a vector bundle $\pi: E \rightarrow M$ is a *superbundle* if it is the direct sum $\pi = \pi^0 \oplus \pi^1$ of two vector bundles $\pi^0: E_0 \rightarrow M$ and $\pi^1: E_1 \rightarrow M$.

Recall that each vector bundle α over M determines a supermanifold as follows [1, 10, 11]. The underlying even manifold is M , and the structure sheaf is the sheaf of sections of the exterior algebra of α^* .

Denote by $\alpha_{k,l}: J^k(\alpha) \rightarrow J^l(\alpha)$ and $\alpha_k: J^k(\alpha) \rightarrow M$ the ordinary projections of jet spaces.

Consider the pullback $(\pi_k^0)^*(\pi_k^1): J^k(\pi^0) \times_M J^k(\pi^1) \rightarrow J^k(\pi^0)$ of the bundle $\pi_k^1: J^k(\pi^1) \rightarrow M$ along the projection $\pi_k^0: J^k(\pi^0) \rightarrow M$. We define the *k-jet space* of π to be the supermanifold that corresponds to the bundle $(\pi_k^0)^*(\pi_k^1)$. It will be denoted by $J^k(\pi)$. In particular, the underlying even manifold of $J^k(\pi)$ is $J^k(\pi^0)$. The natural projections $\pi_k: J^k(\pi) \rightarrow M$ and $\pi_{k,l}: J^k(\pi) \rightarrow J^l(\pi)$ for $k > l$ yield a chain whose inverse limit is said to be the *infinite jet space* and is denoted by $J^\infty(\pi)$.

Let x^1, \dots, x^n be local coordinates in M , u^1, \dots, u^{m_0} and u^{m_0+1}, \dots, u^m be local fiber coordinates in E_0 and E_1 respectively. Then u_σ^j are local derivative coordinates on $J^k(\pi)$. The coordinates x^i and u_σ^j for $j \leq m_0$ are even, while u_σ^j for $j > m_0$ are odd.

The superalgebra $\mathcal{F}(\pi)$ of smooth functions on $J^\infty(\pi)$ is defined as the direct limit of the chain of injections $\pi_{k,k-1}^*$. The $\mathcal{F}(\pi)$ -module $\Lambda^*(\pi)$ of differential forms on $J^\infty(\pi)$ is defined in the similar way. Below we omit the letter π and write simply \mathcal{F} , Λ^k , and so on, when no confusion can arise.

A *horizontal module* is the \mathcal{F} -module of sections of $\pi_\infty^*(\alpha)$, where α is a superbundle over M . Denote by \varkappa the horizontal module corresponding to the bundle π itself.

Let P_1 and P_2 be horizontal modules of even vector bundles. A linear differential operator $\Delta: P_1 \rightarrow P_2$ is called *C-differential* if it can be restricted to the graphs of all infinitely prolonged sections of the bundle. The set of all *C-differential* operators from P_1 to P_2 is denoted by $\mathcal{C}\text{Diff}(P_1, P_2)$. This definition can be generalized to the case of superbundles [5]. In coordinates, *C-differential* operators have the form of a matrix $(a_{ij}^\sigma D_\sigma)$, where $a_{ij}^\sigma \in \mathcal{F}$, $D_\sigma = D_{i_1} \circ \dots \circ D_{i_r}$ for $\sigma = i_1 \dots i_r$ and D_i is the total derivative operator with respect to x^i .

A π_∞ -vertical vector field on $J^\infty(\pi)$ is called an *evolutionary field* if it commutes with all D_i (this property does not depend on the choice of coordinates). In coordinates, each evolutionary field is of the form $E_\varphi = \sum_{j,\sigma} D_\sigma(\varphi^j) \partial / \partial u_\sigma^j$, where $\varphi^j \in \mathcal{F}$.

Let P be a horizontal module. For each element $p \in P$ there is a *C-differential* operator $\ell_p: \varkappa \rightarrow P$ called the *universal linearization* of p and defined by $\ell_p(\varphi) =$

$(-1)^{p\varphi} E_\varphi(p)$, with $\varphi \in \mathfrak{z}$. Here and subsequently, symbols used as the exponents of (-1) stand for the corresponding parities.

A differential form $\omega \in \Lambda^k$ on $J^\infty(\pi)$ is called a *Cartan form* if its pull-back through any prolonged section vanishes. In coordinates, Cartan forms contain factors of the type $\omega_\sigma^j = du_\sigma^j - \sum_i u_{\sigma i}^j dx^i$. Denote the module of all Cartan q -forms by $\mathcal{C}\Lambda^q$. Then $d(\mathcal{C}\Lambda^q) \subset \mathcal{C}\Lambda^{q+1}$. Therefore the quotient \bar{d} of d , acting on $\bar{\Lambda}^* = \Lambda^*/\mathcal{C}\Lambda^*$, is well defined. Elements of $\bar{\Lambda}^q$ are called *horizontal forms*. In coordinates, $\bar{\Lambda}^q$ is generated by $f dx^{i_1} \wedge \cdots \wedge dx^{i_q}$, where $f \in \mathcal{F}$, and $\bar{d}(f dx^{i_1} \wedge \cdots \wedge dx^{i_q}) = \sum_{i=1}^n D_i(f) dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_q}$. The cohomology $\bar{H}^i(\pi)$ of the complex $(\bar{\Lambda}^*, \bar{d})$ coincides with the de Rham cohomology of M for all degrees i up to $n-1$ [12, 13].

If P is an \mathcal{F} -module, we write $\hat{P} = \text{Hom}_{\mathcal{F}}(P, \bar{\Lambda}^n)$ and consider the natural pairing $\langle \cdot, \cdot \rangle: \hat{P} \times P \rightarrow \bar{\Lambda}^n$. We recall that for each operator $\Delta \in \mathcal{C}\text{Diff}(P, Q)$ there exists a unique operator $\Delta^* \in \mathcal{C}\text{Diff}(\hat{Q}, \hat{P})$ such that

$$\langle [\hat{q}, \Delta(p)] \rangle = (-1)^{\Delta \hat{q}} [\langle \Delta^*(\hat{q}), p \rangle], \quad \hat{q} \in \hat{Q}, \quad p \in P,$$

where $[\omega]$ denotes the horizontal cohomology class of $\omega \in \bar{\Lambda}^n$. The operator Δ^* is called *adjoint* to Δ . In coordinates, $(\sum_\sigma a_{ij}^\sigma D_\sigma)^* = (\sum_\sigma (-1)^{|\sigma|} D_\sigma \circ a_{ij}^\sigma)^{\text{st}}$, where $a_{ij}^\sigma \in \mathcal{F}$ and the symbol ‘st’ denotes the supertransposition.

Since evolutionary fields commute with \bar{d} , the cohomology class $[E_\varphi(\omega)]$ for $\omega \in \bar{\Lambda}^n$ is well defined; denote it by $E_\varphi([\omega])$. We have $E_\varphi([\omega]) = [\langle \varphi, \mathcal{E}(\omega) \rangle]$, where $\mathcal{E}(\omega) = \ell_\omega^*(1)$ is the *Euler operator*. In coordinates, $\mathcal{E}(L dx^1 \wedge \cdots \wedge dx^n) = (\delta L / \delta u^j)$, where $\delta L / \delta u^j = \sum_\sigma (-1)^{|\sigma|} D_\sigma(\partial L / \partial u_\sigma^j)$.

Variational Poisson bracket. In this section we use the general construction of Section 1 to introduce a (super)symmetric bracket on multilinear maps of the form $\bar{H}^n \times \cdots \times \bar{H}^n \rightarrow \bar{H}^n$. Then, we provide a description of maps and the bracket in terms of selfadjoint operators. To this aim, we take $G = \mathbb{Z}_2$ and $V = \bar{H}^n$. The space $\mathcal{M}_k(\bar{H}^n)$ is defined to be the set of multilinear maps f of the form

$$f([\omega_1], \dots, [\omega_k]) = [\tilde{f}(\omega_1, \dots, \omega_k)]$$

where $\tilde{f}: \bar{\Lambda}^n \times \cdots \times \bar{\Lambda}^n \rightarrow \bar{\Lambda}^n$ is a multilinear differential operator. It is easily shown that we can take \mathfrak{z} for the algebra \mathfrak{g} . The spaces $\mathfrak{g}^{(k)}$, thus constructed, are denoted by $\mathfrak{z}^{(k)}$. Elements of $\mathfrak{z}^{(k)}$ will be referred to as *variational (super)symmetric multivectors*. The bracket $[\cdot, \cdot]$ on $\mathfrak{z}^{(k)}$ defined in Section 1 is called the *variational Poisson bracket*.

Let us denote by $\mathcal{C}\text{Diff}_{(k)}^{\text{self}}(P, \hat{P})$ the module of k -linear \mathcal{C} -differential operators $\Delta: P \times \cdots \times P \rightarrow \hat{P}$ which are (graded) symmetric and selfadjoint in each argument.

Theorem 2 ([5]). *For each $f \in \mathfrak{z}^{(k)}$ there exists a unique \mathcal{C} -differential operator $\Delta_f \in \mathcal{C}\text{Diff}_{(k-1)}^{\text{self}}(\hat{\mathfrak{z}}, \mathfrak{z})$ such that*

$$f(\omega_1, \dots, \omega_k) = [\langle \Delta_f(\mathcal{E}(\omega_1), \dots, \mathcal{E}(\omega_{k-1})), \mathcal{E}(\omega_k) \rangle].$$

From now on, we identify variational multivectors with the corresponding \mathcal{C} -differential operators.

Theorem 3 ([5]). *If $\Delta_1 \in \mathfrak{X}^{(k)}$ and $\Delta_2 \in \mathfrak{X}^{(l)}$ then for all $\psi_1, \dots, \psi_{k+l-2} \in \hat{\mathfrak{X}}$ we have*

$$\begin{aligned}
(1) \quad & \llbracket \Delta_1, \Delta_2 \rrbracket(\psi_1, \dots, \psi_{k+l-2}) \\
&= (-1)^{\Delta_1 \Delta_2} \left(\sum_{\sigma \in S_{k+l-2}^{l-1}} \epsilon_\psi(\sigma) (-1)^{\Delta_1 \psi_{\sigma(1, l-1)}} \ell_{\Delta_2, \psi_{\sigma(1, l-1)}}(\Delta_1(\psi_{\sigma(l, k+l-2)})) \right. \\
&\quad - \sum_{\sigma \in S_{k+l-2}^k} \epsilon_\psi(\sigma) \Delta_2(\ell_{\Delta_1, \psi_{\sigma(1, k-1)}}^*(\psi_{\sigma(k)}, \psi_{\sigma(k+1, k+l-2)})) \\
&\quad - \sum_{\sigma \in S_{k+l-2}^{k-1}} \epsilon_\psi(\sigma) (-1)^{\Delta_2 \psi_{\sigma(1, k-1)}} \ell_{\Delta_1, \psi_{\sigma(1, k-1)}}(\Delta_2(\psi_{\sigma(k, k+l-2)})) \\
&\quad \left. + \sum_{\sigma \in S_{k+l-2}^l} \epsilon_\psi(\sigma) \Delta_1(\ell_{\Delta_2, \psi_{\sigma(1, l-1)}}^*(\psi_{\sigma(l)}, \psi_{\sigma(l+1, k+l-2)})) \right),
\end{aligned}$$

where $\ell_{\Delta, \psi_1, \dots, \psi_s}(\varphi) = (-1)^{(\psi_1 + \dots + \psi_s)\varphi} \ell_\Delta(\varphi)(\psi_1, \dots, \psi_s)$, for $\varphi \in \mathfrak{X}$, and $\psi_{\sigma(k_1, k_2)}$ is the notation for the vector $\psi_{\sigma(k_1)}, \dots, \psi_{\sigma(k_2)}$. When $\psi_{\sigma(k_1, k_2)}$ is used as the exponent of (-1) it means the sum of degrees of $\psi_{\sigma(k_1)}, \dots, \psi_{\sigma(k_2)}$.

3 Examples and applications

Cotangent bundle to a bundle and the variational Poisson bracket. Here we compare our variational Poisson brackets with the Poisson brackets constructed by Kupershmidt [8]. Let us consider the bundles $\hat{\pi}: \hat{E} = E^* \otimes_M \bigwedge^n(T^*M) \rightarrow M$ and $\Pi = \pi \oplus \hat{\pi}$. Following Kupershmidt [8] we call the bundle $\Pi_\infty: J^\infty(\Pi) \rightarrow M$ the *cotangent bundle* to the bundle π .

Let us denote by p^j , $j = 1, \dots, m$, the fiber coordinates in \hat{E} dual to u^j with respect to a volume form on M . Then coordinates in $J^\infty(\Pi)$ are x^i , u_σ^j , p_σ^j .

We see that $\mathfrak{X}(\Pi) = \mathfrak{X}_\Pi \oplus \hat{\mathfrak{X}}_\Pi$, where $\mathfrak{X}_\Pi = \Gamma(\Pi^*(\pi))$. On the space $\bar{H}^n(\Pi)$ there exists a natural *Poisson bracket* [8] $\llbracket F, H \rrbracket = [\langle \mathcal{E}(F), A(\mathcal{E}(H)) \rangle]$, for $F, H \in \bar{H}^n(\Pi)$, where $A: \hat{\mathfrak{X}}(\Pi) \rightarrow \mathfrak{X}(\Pi)$, $A(\psi, \varphi) = (-\varphi, \psi)$ for $\varphi \in \mathfrak{X}_\Pi$ and $\psi \in \hat{\mathfrak{X}}_\Pi$. In coordinates we have

$$\llbracket F, H \rrbracket = \sum_j \left((-1)^{p^j(F+1)} \frac{\delta F}{\delta p^j} \frac{\delta H}{\delta u^j} - (-1)^{u^j F} \frac{\delta F}{\delta u^j} \frac{\delta H}{\delta p^j} \right).$$

Now, since by definition elements of $\mathcal{F}(\hat{\pi})$ are identified with differential operators from $\Gamma(\hat{\pi})$ to $C^\infty(M)$, we have the natural inclusion $\mathcal{C}\text{Diff}(\hat{\mathfrak{X}}, \mathcal{F}) \rightarrow \mathcal{F}(\Pi)$, which uniquely prolongs to the inclusion of algebras $\mathcal{C}\text{Diff}_{(*)}^{\text{sym}}(\hat{\mathfrak{X}}, \mathcal{F}) \rightarrow \mathcal{F}(\Pi)$. This leads to the further inclusions $\mathcal{C}\text{Diff}_{(*)}^{\text{sym}}(\hat{\mathfrak{X}}, \bar{\Lambda}^n) \rightarrow \bar{\Lambda}^n(\Pi)$ and $\mathfrak{X}^{(*)} \rightarrow \bar{H}^n(\Pi)$, and to the following theorem (see [5] for details and proofs).

Theorem 4 ([5]). *The above Poisson bracket extends the variational Poisson bracket to $\bar{H}^n(\Pi)$.*

Variational Schouten bracket (antibracket) and Hamiltonian formalism.

Using an appropriate commutation factor, one easily obtains odd counterparts of the constructions from the previous subsection: super skew-symmetric multivectors and the variational Schouten bracket. This bracket in coordinates coincides with the *antibracket* [4], thus, our results provide a geometrical description of the antibracket.

Let us denote by $\mathcal{CDiff}_{(k)}^{\text{skew}}(\mathcal{X}, \hat{\mathcal{X}})$ the module of k -linear \mathcal{C} -differential operators $\Delta: \mathcal{X} \times \cdots \times \mathcal{X} \rightarrow \hat{\mathcal{X}}$ which are skew-symmetric and skew-adjoint in each argument. By analogy with Theorem 2 it can be proved that each super skew-symmetric multivector can be represented through a unique \mathcal{C} -differential operator $\Delta \in \mathcal{CDiff}_{(k-1)}^{\text{skew}}(\hat{\mathcal{X}}, \mathcal{X})$. Of course, formula (1) holds for the variational Schouten bracket too, *mutatis mutandis*.

The variational Schouten bracket is useful in the Hamiltonian theory of evolution equation. Let us consider a purely even bundle π for the moment. An operator $A \in \mathcal{CDiff}_{(1)}^{\text{skew}}(\hat{\mathcal{X}}, \mathcal{X})$ (variational bivector) is called *Hamiltonian* if $\llbracket A, A \rrbracket = 0$. A Hamiltonian operator defines a Lie algebra structure on $\bar{H}^n(\pi)$. An evolution equation $u_t = f$, $f \in \mathcal{X}$ is said to be *Hamiltonian* with respect to A if $A_t - \llbracket A, f \rrbracket = 0$.

It can be shown that an evolution equation is Hamiltonian if and only if $\ell \circ A + A \circ \ell^* = 0$, where ℓ is the linearization of the equation at hand. Combining this with other geometric considerations, one obtains an efficient method to find Hamiltonian structures for a given equation [6].

Local variational differential operators. We want to identify the space \bar{H}^n with a subspace of functionals on $\Gamma(\pi)$. Consider first the case of a purely even vector bundle $\pi: E \rightarrow M$.

A *domain* is an oriented open subset U of M with compact closure \bar{U} and smooth boundary ∂U . Let $U_1, U_2 \subset M$ be two domains such that $B = \partial U_1 = \partial U_2$ and the orientations of U_1, U_2 induce opposite orientations on B . Let $s_i \in \Gamma(\pi, \bar{U}_i)$, $i = 1, 2$, $[s_1]_x^\infty = [s_2]_x^\infty$ for all $x \in B$. Here and below $[s_i]_x^\infty$ is the infinite jet of s_i at x . A horizontal cohomology class $[\omega] \in \bar{H}^n$ gives a well-defined functional on such 4-tuples (U_1, U_2, s_1, s_2) as follows

$$\omega(U_1, U_2, s_1, s_2) = \int_{\bar{U}_1} j_\infty(s_1)^*(\omega) + \int_{\bar{U}_2} j_\infty(s_2)^*(\omega),$$

where $j_\infty(s_i)$ is the section $x \mapsto [s_i]_x^\infty$ of π_∞ . It is easily seen that each nonzero class determines a nonzero functional.

Following [14] we call functionals of the form $F(\omega_1, \dots, \omega_N)$, where F is a smooth function in many arguments, *multilocal functionals*. Denote by $\mathcal{A} = \mathcal{A}(\pi)$ the algebra of multilocal functionals. A natural question arises: what are the relations in \mathcal{A} , i.e., for what nonzero smooth functions F and horizontal forms ω_i the expression

$F(\omega_1, \dots, \omega_N)$ induces an identically zero functional? It turns out that at least locally (in a certain sense) all relations in \mathcal{A} are generated by linear relations in \bar{H}^n .

Let $U \subset M$ be a domain. For a finite subset $V \subset U$ and $s_0 \in \Gamma(\pi, \bar{U})$ put $\Gamma_{s_0, V} = \{s \in \Gamma(\pi, \bar{U}) \mid [s]_x^\infty = [s_0]_x^\infty \ \forall x \in V \cup \partial U\}$. Such subsets of $\Gamma(\pi, \bar{U})$ are endowed with the C^∞ -topology.

Theorem 5 ([5]). *Let $s_0 \in \Gamma(\pi, \bar{U})$, V be a finite subset of U , and $\omega_1, \dots, \omega_l \in \bar{\Lambda}^n$. Consider the linear map $\Psi: \Gamma(\pi) \rightarrow \mathbb{R}^l$, $\Psi(s) = (\int_{\bar{U}} j_\infty(s)^*(\omega_1), \dots, \int_{\bar{U}} j_\infty(s)^*(\omega_l))$. Suppose that there exists a neighborhood $\Gamma \subset \Gamma_{s_0, V}$ of s_0 such that $\Psi(\Gamma)$ does not contain any open subset of \mathbb{R}^l . Then there is a nontrivial linear combination $a_1\omega_1 + \dots + a_l\omega_l$ that induces a constant functional on a neighborhood $\Gamma' \subset \Gamma_{s_0, V \cup V'}$ of s_0 , where $V' \subset U$ consists of $k < l$ points.*

Return to a 4-tuple $T = (U_1, U_2, s_1, s_2)$. For finite subsets $V_i \subset U_i$, $i = 1, 2$, set $\Gamma(V_1, V_2) = \{(U_1, U_2, s'_1, s'_2) \mid s'_i \in \Gamma_{s_i, V_i}, i = 1, 2\}$. Below a *weak neighborhood* of T is a C^∞ -topological neighborhood of T in the set $\Gamma(V_1, V_2)$ for some V_1, V_2 .

A smooth function $F: \mathbb{R}^N \rightarrow \mathbb{R}$ is said to be *almost identically zero* around $x \in \mathbb{R}^N$ if for any neighborhood U of x there is a nonempty open subset $U' \subset U$ such that $F|_{U'} = 0$. In particular, in this case all partial derivatives of F vanish at x .

Theorem 6 ([5]). *Suppose that a multilocal functional $F(\omega_1, \dots, \omega_N)$ is zero on some weak neighborhood of T and the function F is not almost identically zero around $(\omega_1(T), \dots, \omega_N(T)) \in \mathbb{R}^N$. Then there is a nontrivial linear combination $a_1\omega_1 + \dots + a_N\omega_N$ that induces a constant functional on a weak neighborhood of T .*

Corollary 1 ([5]). *For any variational multivector $f \in \mathcal{X}^{(k)}$ the formula*

$$(2) \quad \nabla_f(F(\omega_1, \dots, \omega_N)) = \sum_{1 \leq i_1, \dots, i_k \leq N} \frac{\partial^k F}{\partial t_{i_1} \dots \partial t_{i_k}}(\omega_1, \dots, \omega_N) \cdot f([\omega_{i_1}], \dots, [\omega_{i_k}]).$$

determines a well-defined differential operator $\nabla_f: \mathcal{A} \rightarrow \mathcal{A}$ of order k .

Following [14], we call (2) *local variational differential operators*.

If π is a vector superbundle then the horizontal cohomology space is \mathbb{Z} -graded $\bar{H}^n = \bigoplus_{i \geq 0} H_i^n$, the space H_0^n being isomorphic to the n -th horizontal cohomology space of the even component π^0 . In this case we define $\mathcal{A}(\pi) = \mathcal{A}(\pi^0) \otimes_{\mathbb{R}} S(\bigoplus_{i \geq 1} H_i^n)$. Applying Corollary 1 to π^0 , one can show that local variational differential operators are well defined on $\mathcal{A}(\pi)$.

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