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Quantum structures in general relativistic theories

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Abstract

In this work we formulate quantum structures on an Einstein general relativistic background and present an existence and classification theorem. This approach follows the scheme of the Galilei general relativistic quantum mechanics, as formulated by Jadczyk and Modugno. The existence and classification result is analogous to that of geometric quantisation.

Introduction

One of the most important subjects of theoretical physics is the covariant formulation of quantum mechanics.

Several authors[2,9,12,13] have fruitfully investigated a covariant approach to classical and quantum mechanics in a curved Galilei background, i.e. on a curved spacetime with absolute time.

Recently, it has been presented a formulation of Galilei classical and quantum mechanics based on jets, connections and cosymplectic forms[1,4,5]. This formulation has the advantage of being manifestly covariant, due to the use of intrinsic techniques on manifolds. This approach recovers all examples of standard quantum mechanics in the flat case. In particular, the standard examples of geometric quantisation (i.e. harmonic oscillator and hydrogen atom) are recovered in an easier way.

The above formulation can be rephrased in Einstein's general relativity[6,7]. In this paper, after recalling quantum structures in Galilei's case, we introduce the Einstein's general relativistic quantum structures. Moreover, we give a theorem of Kostant–Souriau type[8], which states a topological necessary and sufficient existence condition on the spacetime and the cosymplectic form. Also, we give a classification theorem for quantum structures.

We assume manifolds and maps to be C^∞. Moreover, we assume c to be the *light velocity* and $\hbar$ to be the *Planck's constant*. Finally, we assume a particle mass m and charge q For a rigorous mathematical treatment of units of measurement, see Jadczyk and Modugno[5].

1 Galilei theory

Here, we present a summary of the classical theory and quantum structures[1,4,5], together with the existence and classification results for quantum structures[10].

The *spacetime* is assumed to be a fibred manifold $t : \boldsymbol{E} \to \boldsymbol{T}$, where $\boldsymbol{E}$ is an orientable 4–dimensional manifold and $\boldsymbol{T}$ is a 1–dimensional oriented affine space associated with the real line $I\!R$. We assume a *vertical Riemannian metric* on $\boldsymbol{E}$, $g : \boldsymbol{E} \to V^*\boldsymbol{E} \otimes_{\boldsymbol{E}} V^*\boldsymbol{E}$. The first jet bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$ is said to be the *phase space*.

Spacetime connections are defined to be dt-preserving linear connections on $\boldsymbol{E}$. A spacetime connection K and the metric g induce naturally a 2–form[1] $\Omega[K]$ on $J_1\boldsymbol{E}$ for the coordinate expression. We assume $\boldsymbol{E}$ to be endowed with a spacetime connection $K^\natural$ (the *gravitational field*) such that $d\Omega[K^\natural] = 0$, and a closed 2–form F on $\boldsymbol{E}$ (the *electromagnetic field*). We couple $K^\natural$ and F by considering the sum $\Omega := \Omega[K^\natural] + \frac{q}{2m} F$, so $d\Omega = 0$. This implies that $K^\natural$ is metric, but it is not completely determined by g.

We say the *quantum bundle* to be a complex line–bundle $\boldsymbol{Q} \to \boldsymbol{E}$ endowed with a Hermitian fibre metric h. Moreover, we assume on the bundle $J_1\boldsymbol{E} \times_{\boldsymbol{E}} \boldsymbol{Q} \to J_1\boldsymbol{E}$ a connection Ч, called the *quantum connection*[1], which is Hermitian, *universal*[5] (roughly, it is trivial with respect to the fibring $J_1\boldsymbol{E} \to \boldsymbol{E}$), and such that its curvature fulfills $R[\text{Ч}] = i\frac{m}{\hbar}\Omega$. The pair $(\boldsymbol{Q}, \text{Ч})$ is said to be a *quantum structure*. Two quantum structures are said to be equivalent if there exists an isomorphism of the underlying complex Hermitian line bundles on $\boldsymbol{E}$ which maps one quantum connection into the other.

A covariant formulation[1] of quantum mechanics of a scalar particle in a spacetime with absolute time is made by the choice of a quantum structure. An implementation of the correspondence principle in a covariant formulation involving a curved spacetime is then achieved.

Denote by H the Čech cohomology functor. The abelian group inclusion $i : \mathbb{Z} \to I\!R$ yields an abelian group morphism $i : H^2(\boldsymbol{E}, \mathbb{Z}) \to H^2(\boldsymbol{E}, I\!R)$. It can be proved[10] that the closed form Ω determines a class in $H^2(\boldsymbol{E}, I\!R)$. Moreover, it can be proved[10] that there exists a quantum structure $(\boldsymbol{Q}, \text{Ч})$ if and only if Ω determines a cohomology class in the subgroup $[\Omega] \in i(H^2(\boldsymbol{E}, \mathbb{Z})) \subset H^2(\boldsymbol{E}, I\!R)$. In this case, there exists a bijection between the set of equivalence classes of quantum structures and the cohomology group $H^1(\boldsymbol{E}, U(1))$. Hence, if $\boldsymbol{E}$ is simply connected, then $H^1(\boldsymbol{E}, U(1)) = \{0\}$, so there exists a unique equivalence class of quantum structures[14].

2 Einstein spacetime

In this section, we show that the geometric constructions of classical and quantum Galilei theory can be recovered in Einstein's case[6].

We assume the *spacetime* to be a manifold $\boldsymbol{M}$, with $\dim \boldsymbol{M} = 4$, endowed with a scaled Lorentz metric g whose signature is $(+---)$. Moreover, we assume $\boldsymbol{M}$ to be oriented and time–oriented. We will use charts (x^φ), $0 \le \varphi \le 3)$ on $\boldsymbol{M}$ such that ∂_0 is time–like and time–oriented, and ∂_i $(1 \le i \le 3)$ are space–like. We deal with the first–order jet $U_1\boldsymbol{M}$ of time–like 1–dimensional submanifolds of $\boldsymbol{M}^6$; $U_1\boldsymbol{M}$ is said to be the *phase space*. We have a natural fibring $U_1\boldsymbol{M} \to \boldsymbol{M}$; a typical chart (x^0, x^i) on $\boldsymbol{M}$ induces an adapted chart $(x^0, x^i; x^i_0)$ on $U_1\boldsymbol{M}$. Moreover, the metric g induces naturally a 1–form τ on $U_1\boldsymbol{M}^6$ whose coordinate expression is $\tau^\natural = \alpha/c\,(g_{0\lambda} + g_{i\lambda}x^i_0)d^\lambda$, where $\alpha = \left(\sqrt{g_{00} + 2g_{0j}x^j_0 + g_{ij}x^i_0x^j_0}\right)^{-1}$.

The Levi–Civita connection $K^\natural$ and the metric g induce naturally a 2–form $\Omega[K^\natural]$ on $U_1\boldsymbol{M}$. We have the coordinate expression

$$\Omega^\natural = c\alpha(g_{i\mu} - \tau^\natural_i\tau^\natural_\mu)(d^i_0 - \Gamma_\phi{}^i{}_0 d^\phi) \wedge d^\mu\,, \tag{1}$$

where $\Gamma_\phi{}^i{}_0 = K_\phi{}^i{}_j x^j_0 + K_\phi{}^i{}_0 - x^i_0(K_\varphi{}^0{}_j x^j_0 + K_\varphi{}^0{}_0)$. We can prove that $c^2 d\tau = \Omega[K^\natural]$, hence $d\Omega[K^\natural] = 0$.

We assume $\boldsymbol{M}$ to be endowed with a closed 2–form F (the *electromagnetic field*). We couple $K^\natural$ and F by considering the sum $\Omega := \Omega[K^\natural] + \frac{q}{2mc}F$. We have $d\Omega = 0$.

Now, we develop the geometric structures for the quantisation of the mechanics of one scalar particle in an Einstein general relativistic background. We say the *quantum bundle* to be a complex line–bundle $\boldsymbol{Q} \to \boldsymbol{M}$ endowed with a Hermitian fibre metric h. Moreover, we assume on the bundle $U_1\boldsymbol{M} \times_{\boldsymbol{M}} \boldsymbol{Q} \to U_1\boldsymbol{M}$ a connection Ч, called the *quantum connection*, which is Hermitian, universal, and such that its curvature fulfills $R[\text{Ч}] = i\frac{m}{\hbar}\Omega$. The coordinate expression of Ч turns out to be

$$\text{Ч} = d^\lambda \otimes \partial_\lambda + d^i_0 \otimes \partial^0_i + i\frac{m}{\hbar}\left(\tau_\lambda + \frac{q}{mc}A_\lambda\right)d^\lambda\,, \tag{2}$$

where $A_\lambda d^\lambda$ is a local potential of the electromagnetic field F. The equivalence of quantum structures $(\boldsymbol{Q}, \text{Ч})$ is defined analogously to the Galilei case.

We could proceed by defining an algebra of quantisable functions[7], a quantum Lagrangian and an algebra of quantum operators. This programme will be completed in a future work.

As for the existence and the classification of quantum structures, we can state results analogous to the Galilei case. We note that, in the Einstein case, the cohomology class of Ω depends only on the cohomology class of F.

Theorem. *There exists a quantum structure* $(\boldsymbol{Q}, Ч)$ *if and only if* F *determines a cohomology class in the subgroup* $[F] \in i(H^2(\boldsymbol{M}, \mathbb{Z})) \subset H^2(\boldsymbol{M}, \mathbb{R})$*. In this case, there exists a bijection between the set of equivalence classes of quantum structures and the cohomology group* $H^1(\boldsymbol{M}, U(1))$*.*

Hence, as in the Galilei case, if $\boldsymbol{M}$ is simply connected, then there exists a unique equivalence class of quantum structures.

From a physical viewpoint, it is very interesting to study concrete exact solutions. The following examples are a starting point for an analysis of the classification of quantum structures in exact solutions in Einstein's general relativity.

1. Minkowski spacetime is topologically trivial, hence there exists a unique equivalence class of quantum structures, namely the trivial one.

2. Schwartzschild spacetime has the topology of $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$, hence it is simply connected. Being $F = 0$, there exists only the equivalence class of the trivial quantum structure.

3. Dirac's monopole. We consider the family of magnetic fields F_q parametrised by $q \in \mathbb{R}$ introduced by Dirac[16]; F_q fulfill the integrality condition if and only if $q/(\hbar c) \in \mathbb{Z}$ (Dirac's charge quantisation condition). So, by topological arguments, to any $q/(\hbar c) \in \mathbb{Z}$ there exists one equivalence class of quantum structures, but, in general, this is not the trivial one.

4. The Aharonov–Bohm effect[15] can be modeled on a Minkowski spacetime with a fixed inertial observer and a solenoidal magnetic field. In this case, spacetime without the origin of the magnetic field is no longer simply connected, and there exist infinitely many inequivalent quantum structures.

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