Finite order variational sequences: a short review

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Abstract

Variational sequences are complexes of modules or sheaf sequences in which one of the maps is the Euler–Lagrange operator, *i.e.*, the differential operator taking a Lagrangian into its Euler–Lagrange form. In this review paper we discuss variational sequences on finite order jets, with special emphasis on Krupka’s approach. We also discuss recent results on this topic as well as possible research directions.

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Introduction

In the Seventies, during a process of geometrization of the calculus of variations, it was realized that operations like passing from a Lagrangian to its Euler–Lagrange form were part of a complex, namely, the *variational sequence*. Foundational contributions to variational sequences are in the papers [3, 7, 12, 39, 40, 41, 42, 43, 44, 45].

Among the problems which were solved by the variational sequence was the so-called global inverse problem of the calculus of variations: given a set of Euler–Lagrange equations, the vanishing of Helmholtz conditions is a necessary and sufficient condition for the existence of a local Lagrangian for the given equations; does there exist a global Lagrangian? It was proved that the answer is in the cohomology of the variational sequence. More precisely, the cohomological obstruction for always having a global Lagrangian is the $n+1$-st de Rham cohomology of the space of independent and dependent variables.

The geometric framework for variational sequences is that of jet spaces. Infinite order jet spaces were used as a rule, with the exception of [3]. There are some technical reasons for that choice: the first and most important is that on infinite order jet spaces the contact distribution is integrable and admits an intrinsic direct summand. This fact leads to much simpler computations.

On the other hand, using infinite order jets one simply drops any information on the order of the objects involved in the computations. In this sense, the use of finite order jets can lead to finer results. A first approach in this sense was in [3]. In that paper the finite order variational sequence was truncated after the space of Euler–Lagrange forms. Moreover, in order to obtain the solution of the global inverse problem the authors resorted to infinite order jets. Another approach was through $\mathcal{C}$-spectral sequences in [8, 9]. But it used one conjecture about the structure of contact forms (see Theorem 1.3).

In [23] Krupka proved the above conjecture and was able to give the first formulation of the (long) variational sequence on finite order jets. The formulation was different from both the so-called variational bicomplex [2, 37] and the $\mathcal{C}$-spectral sequence [7, 44]. The idea is rather simple: consider the de Rham complex on jets of order r. Then a subsequence of forms which yield trivial contribution to action-like functionals is defined. The quotient of the former sequence with the latter one yields the finite order variational sequence.

In this paper, after a preliminary section on jet spaces and contact forms, we describe Krupka's finite order variational sequence. In the final section we discuss the state of the research on this topic.

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1 Jet spaces

Manifolds and maps between manifolds are C^∞. All morphisms of fibred manifolds (and hence bundles) will be morphisms over the identity of the base manifold, unless otherwise specified. In particular, when speaking of 'forms' we will always mean 'C^∞ differential forms'.

We recall some basic facts on jet spaces. Our framework is a fibred manifold

$$\pi : Y \to X,$$

with $\dim X = n$, $\dim Y = n + m$, and n, $m \geq 1$. We have the vector subbundle $VY \stackrel{\text{def}}{=} \ker T\pi$ of TY, which is made by vectors which are tangent to the fibres of Y.

For $1 \leq r$, we are concerned with the r-th jet space $J^r\pi$; we also set $J^0\pi \equiv Y$. For $0 \leq s < r$ we recall the natural fibrings

$$\pi_{r,s} : J^r\pi \to J^s\pi, \qquad \pi_r : J^r\pi \to X,$$

and the affine bundle $\pi_{r,r-1} : J^r\pi \to J^{r-1}\pi$ associated with the vector bundle $\odot^r T^*X \otimes_{J^{r-1}\pi} VY \to J^{r-1}\pi$.

Charts on Y adapted to the fibring are denoted by (x^i, y^σ). Latin indices i, j, ... run from 1 to n and label base coordinates, Greek indices σ, τ, ... run from 1 to m and label fibre coordinates, unless otherwise specified. We denote by $(\partial/\partial x^i, \partial/\partial y^\sigma)$ and (dx^i, dy^σ), respectively, the local bases of vector fields and 1-forms on Y induced by an adapted chart.

We denote (symmetrized) multi-indices by capital letters: $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$. We also set $|I| \stackrel{\text{def}}{=} \sum_k i_k$ and $I! \stackrel{\text{def}}{=} \sigma_1! \cdots \sigma_n!$. The sum of a multiindex with a Latin index $I + i$ will denote the sum of I and the multiindex $(0, \ldots, i, 0, \ldots, 0)$, where 1 is at the i-th entry.

The charts induced on $J^r\pi$ are denoted by (x^i, y^σ_I), where $0 \leq |I| \leq r$ and $y^\sigma_0 \stackrel{\text{def}}{=} y^\sigma$. The local vector fields and forms of $J^r\pi$ induced by the fibre coordinates are denoted by $(\partial/\partial y^\sigma_I)$ and (dy^σ_I), $0 \leq |I| \leq r$, $1 \leq i \leq m$, respectively.

An r-th order (ordinary or partial) *differential equation* is, by definition, a submanifold $S \subset J^r\pi$.

We denote by $j_r s \colon X \to J^r\pi$ the jet prolongation of a section $s \colon X \to Y$ and by $J^r f \colon J^r\pi \to J^r\pi$ the jet prolongation of a fibred morphism $f \colon Y \to Y$ over a diffeomorphism $\bar{f} \colon X \to X$. Any vector field $\xi \colon Y \to TY$ which projects onto a vector field $\xi \colon X \to TX$ can be prolonged to a vector field $\xi^r \colon J^r\pi \to TJ^r\pi$ by prolonging its flow; its coordinate expression is well-known (see, *e.g.*, [5, 37]).

The fundamental geometric structure on jets is the *contact distribution* (or *Cartan distribution*) $C^r \subset TJ^r\pi$. It is the distribution on $J^r\pi$ generated by all vectors which are tangent to the image $j_r s(X) \subset J^r\pi$ of a prolonged section $j_r s$. It is locally generated by the vector fields

$$D_i = \frac{\partial}{\partial x^i} + y^\sigma_{I+i}\frac{\partial}{\partial y^\sigma_I}, \qquad \frac{\partial}{\partial y^\sigma_J}, \tag{1}$$

with $0 \leq |I| \leq r-1$, $|J| = r$. It is easy to show that this distribution is not involutive and does not admit any natural direct summand that complement it to $TJ^r\pi$. While the contact distribution has an essential importance in the symmetry analysis of PDE [5], in this context the dual concept of contact differential forms plays a central role.

Let us denote by $\mathcal{F}_r$ the sheaf of smooth functions on $J^r\pi$.

We denote by Ω^k_r the sheaf of k-forms on $J^r\pi$.

We denote by Ω^*_r the sheaf of forms of any degree on $J^r\pi$.

1.1 Definition. We say that a form $\alpha \in \Omega^k_r$ is a *contact k-form* if

$$(j_r s)^*\alpha = 0$$

for all sections s of π.

We denote by $\mathcal{C}^1\Omega^k_r$ the sheaf of contact k-forms on $J^r\pi$.

We denote by $\mathcal{C}^1\Omega^*_r$ the sheaf of contact forms of any degree on $J^r\pi$.

Note that if $k > n$ then every form is contact, *i.e.*, $\mathcal{C}^1\Omega^k_r = \Omega^k_r$.

It is obvious from the commutation of d and pull-back that $d\mathcal{C}^1\Omega^k_r \subset \mathcal{C}^1\Omega^{k+1}_r$. Moreover, it is obvious that $\mathcal{C}^1\Omega^*_r$ is a sheaf of ideals (with respect to the exterior product) in Ω^*_r. Unfortunately, $\mathcal{C}^1\Omega^*_r$ does not coincide with the ideal generated by 1-forms which annihilate the contact distribution (for this would contradict the non-integrability). More precisely, the following lemma can be easily proved (see, *e.g.*, [23]).

1.2 Lemma. *The sheaf $\mathcal{C}^1\Omega^1_r$ is locally generated (on $\mathcal{F}_r$) by the 1-forms*

$$\omega^\sigma_I \overset{def}{=} dy^\sigma_I - y^\sigma_{I+i}dx^i, \quad 0 \leq |I| \leq r-1.$$

The above differential forms generate an ideal of Ω^*_r. However, such an ideal is not differential, hence it does not coincide with $\mathcal{C}^1\Omega^*_r$. To realize it, the following formula can be easily proved

$$d\omega^\sigma_I = -\omega^\sigma_{I+i} \wedge dx^i, \tag{2}$$

from which it follows that, when $|I| = r-1$, then $d\omega^\sigma_I$, which is a contact 2-form, cannot be expressed through the 1-forms of lemma 1.2 because ω^σ_{I+i} contains derivatives of order $r+1$.

The following theorem is an important achievement by Krupka. It has been first conjectured in [9] ($\mathcal{C}^1\Omega$-hypothesis), then proved in [23, 24].

1.3 Theorem. *Let $k \geq 2$. The sheaf $\mathcal{C}^1\Omega^k_r$ is locally generated (on $\mathcal{F}_r$) by the forms*

$$\omega^\sigma_I, \quad d\omega^\sigma_J, \qquad 0 \leq |I| \leq r-1, \quad |J| = r-1.$$

We can consider forms which are generated by p-th exterior powers of contact forms. More precisely, we have the following definition.

1.4 Definition. Let $p \geq 1$. We say that a form $\alpha \in \Omega^k_r$ is a *p-contact k-form* if it is generated by p-th exterior powers of contact forms.

We denote by $\mathcal{C}^p\Omega^k_r$ the sheaf of p-contact k-forms on $J^r\pi$.

We denote by $\mathcal{C}^p\Omega^*_r$ the sheaf of p-contact forms of any degree on $J^r\pi$.

Finally, we set $\mathcal{C}^0\Omega^*_r \overset{\text{def}}{=} \Omega^*_r$.

In other words, $\mathcal{C}^p\Omega^*_r$ is the p-th power of the ideal $\mathcal{C}^1\Omega^*_r$ in Ω^*_r. Of course, a 1-contact form is just a contact form. We have the obvious inclusion

$$\mathcal{C}^{p+1}\Omega^*_r \subset \mathcal{C}^p\Omega^*_r.$$

It follows that $\mathcal{C}^{p+1}\Omega^*_r$ is a sheaf of ideals of $\mathcal{C}^p\Omega^*_r$, hence of Ω^*_r. Moreover, $d\mathcal{C}^{p+1}\Omega^*_r \subset \mathcal{C}^{p+1}\Omega^*_r$.

Now, we would like to introduce a tool to extract from a form $\alpha \in \Omega^k_r$ the non-trivial part (to the purposes of calculus of variations). In other words, we would like to introduce a map whose kernel is precisely the set of contact forms. Such forms yield no contribution to action-like functionals (see Remark 1.10). First of all, we observe that eq. (2) and Theorem 1.3 suggest that such a map can be constructed if we allow it to increase the jet order by 1. More precisely, it can be easily proved that the contact 1-forms ω^σ_I, with $0 \leq |I| \leq r-1$ generate a natural subbundle $C^*_r \subset J^r\pi \times_{J^{r-1}\pi} T^*J^{r-1}\pi \subset T^*J^r\pi$ [46]. We have the following lemma (see [32, 37]).

1.5 Lemma. *We have the splitting*

$$J^{r+1}\pi \underset{J^r\pi}{\times} T^*J^r\pi = \left(J^{r+1}\pi \underset{X}{\times} T^*X\right) \underset{J^{r+1}\pi}{\oplus} C^*_{r+1}, \tag{3}$$

with projections

$$D^{r+1} \colon J^{r+1}\pi \to T^*X \underset{X}{\otimes} TJ^r\pi, \quad \omega^{r+1} \colon J^{r+1}\pi \to T^*J^r\pi \underset{J^r\pi}{\otimes} VJ^r\pi,$$

and coordinate expression

$$D^{r+1} = dx^i \otimes D_i = dx^i \otimes \left(\frac{\partial}{\partial x^i} + y^\sigma_{I+i}\frac{\partial}{\partial y^\sigma_I}\right),$$
$$\omega^{r+1} = \omega^\sigma_I \otimes \frac{\partial}{\partial y^\sigma_I} = (dy^\sigma_I - y^\sigma_{I+i}dx^i) \otimes \frac{\partial}{\partial y^\sigma_I}.$$

Note that the above construction makes sense through the natural inclusions $VJ^r\pi \subset TJ^r\pi$ and $J^{r+1}\pi \times_X T^*X \subset J^{r+1}\pi \times_{J^r\pi} T^*J^r\pi$, the latter being provided by $T^*\pi_r$.

From elementary multilinear algebra it turns out that we have the splitting

$$J^{r+1}\pi \times_{J^r\pi} \wedge^k T^*J^r\pi = \bigoplus_{p+q=k} \left(J^{r+1}\pi \underset{X}{\times} \wedge^q T^*X\right) \underset{J^{r+1}\pi}{\oplus} \wedge^p C^*_{r+1}.$$

Now, we observe that a form $\alpha \in \Omega^k_r$ fulfills

$$\pi^*_{r+1,r}(\alpha) \colon J^{r+1}\pi \to \wedge^k T^*J^r\pi \subset \wedge^k T^*J^{r+1}\pi,$$

where the inclusion is realized through the map $T^*\pi_{r+1,r}$. Hence, $\pi^*_{r+1,r}(\alpha)$ can be split into $k+1$ factors which, respectively, have 0 contact factors, 1 contact factor, ..., k contact factors. More precisely, let us denote by $\mathcal{H}^q_r$ the set of q-forms of the type

$$\alpha\colon J^r\pi \to \wedge^q T^*X.$$

We have the following proposition (for a proof, see [23, 46, 48]).

1.6 Proposition. *We have the natural decomposition*

$$\pi^*_{r+1,r}(\Omega^k_r) \subset \bigoplus_{p+q=k} \mathcal{C}^p\Omega^p_{r+1} \wedge \mathcal{H}^q_{r+1},$$

with splitting projections $\mathrm{pr}^{p,q}\colon \Omega^k_r \to \mathcal{C}^p\Omega^p_{r+1} \wedge \mathcal{H}^q_{r+1}$ *defined by*

$$\mathrm{pr}^{p,q}(\alpha) = \left(\binom{p+q}{q} \odot^p i_{D^{r+1}} \odot \odot^q i_{\omega^{r+1}}\right) \circ \pi^*_{r+1,r},$$

where $i_{D^{r+1}}$, $i_{\omega^{r+1}}$ *stand for contractions followed by a wedge product.*

Note that the above maps $\mathrm{pr}^{p,q}$ are not surjective. See [46] for more details.

1.7 Definition. We say the *horizontalization* to be the map

$$h^{p,q}\colon \mathcal{C}^p\Omega^{p+q}_r \to \mathcal{C}^p\Omega^p_{r+1} \wedge \mathcal{H}^q_{r+1}, \quad \alpha \mapsto \mathrm{pr}^{p,q}(\alpha).$$

We denote by

$$\overline{\Omega}^{p,q}_r \stackrel{\text{def}}{=} h^{p,q}(\mathcal{C}^p\Omega^{p+q}_r) \tag{4}$$

the image of the horizontalization; we say an element $\bar{\alpha} \in \overline{\Omega}^{0,q}_r$ to be a *horizontal form.*

Probably the first occurrence of horizontalization is in [22]. Of course, horizontalization is just the projection on forms which have no contact factors. Note that, if $q > n$, then horizontalization is the zero map. In coordinates, if $0 < q \le n$, then

$$\alpha = \alpha^{I_1\cdots I_h}_{\sigma_1\cdots\sigma_h i_{h+1}\cdots i_q}\, dy^{\sigma_1}_{I_1} \wedge \cdots \wedge dy^{\sigma_h}_{I_h} \wedge dx^{i_{h+1}} \wedge \cdots \wedge dx^{i_q}$$

and the coordinate expression of the horizontalization is

$$h^{0,q}(\alpha) = y^{\sigma_1}_{I_1+i_1} \cdots y^{\sigma_h}_{I_h+i_h} \alpha^{I_1\cdots I_h}_{\sigma_1\cdots\sigma_h i_{h+1}\cdots i_q}\, dx^{i_1} \wedge \cdots \wedge dx^{i_q}, \tag{5}$$

where $0 \le h \le q$. The coordinate expressions of $h^{p,q}$ can be obtained in a similar way (see [3, 23, 24, 46]).

Note that if $n > 1$ then the above form is not the most general polynomial in $(r+1)$-st derivatives, even if $q = 1$. For $q > 1$ the skew-symmetrization in the indexes i_1,..., i_h yields a peculiar structure in the polynomial, in which the sums of all terms of the same degree are said to be *hyperjacobians.* Finally, we observe that if $n = 1$ then the horizontalization is surjective on the space of forms with affine coefficients with respect to $r+1$-st derivatives [25].

The technical importance of horizontalization is in the next two results.

1.8 Lemma. *Let $\alpha \in \Omega^q_r$, with $0 \le q \le n$, and $s\colon X \to Y$ be a section. Then*

$$(j_r s)^*(\alpha) = (j_{r+1} s)^*(h^{0,q}(\alpha))$$

1.9 Proposition. *Let $p \ge 0$. The kernel of $h^{p,q}$ coincides with $p+1$-contact q-forms,* i.e.,

$$\mathcal{C}^{p+1}\Omega^q = \ker h^{p,q}.$$

For a proof of both results, see, for example, [48].

The above decomposition also affects the exterior differential. Namely, the pull-back of the differential can be split in two operators, one of which raises the contact degree by one, and the other raises the horizontal degree by one. More precisely, in view of proposition 1.6 and following [37], we introduce the maps

$$i_H\colon \Omega^k_r \to \Omega^k_{r+1}, \quad i_H = i_{D^{r+1}} \circ \pi^*_{r+1,r}, \tag{6a}$$

$$i_V\colon \Omega^k_r \to \Omega^k_{r+1}, \quad i_V = i_{\omega^{r+1}} \circ \pi^*_{r+1,r}. \tag{6b}$$

The maps i_H and i_V are two derivations along $\pi_{r+1,r}$ of degree 0. Together with the exterior differential d they yield two derivations along $\pi_{r+1,r}$ of degree 1, the *horizontal* and *vertical differential*

$$d_H \overset{\text{def}}{=} i_H \circ d - d \circ i_H : \Omega^k_r \to \Omega^k_{r+1},$$
$$d_V \overset{\text{def}}{=} i_V \circ d - d \circ i_V : \Omega^k_r \to \Omega^k_{r+1},$$

It can be proved (see [37]) that d_H and d_V fulfill the properties

$$d_H^2 = d_V^2 = 0, \qquad d_H \circ d_V + d_V \circ d_H = 0, \tag{7a}$$

$$d_H + d_V = (\pi^{r+1}_r)^* \circ d, \tag{7b}$$

$$(j_{r+1}s)^* \circ d_V = 0, \qquad d \circ (j_r s)^* = (j_{r+1}s)^* \circ d_H. \tag{7c}$$

The action of d_H and d_V on functions $f : J^rY \to \mathbb{R}$ and one–forms on J^rY uniquely characterizes d_H and d_V. We have the coordinate expressions

$$d_H f = D_i f \, dx^i = \left(\frac{\partial f}{\partial x^i} + y^\sigma_{I+i} \frac{\partial f}{\partial y^\sigma_I} \right) dx^i, \tag{8a}$$

$$d_H dx^i = 0, \qquad d_H dy^\sigma_I = -dy^\sigma_{I+i} \wedge dx^i, \qquad d_H \omega^\sigma_I = -\omega^\sigma_{I+i} \wedge dx^i, \tag{8b}$$

$$d_V f = \frac{\partial f}{\partial y^\sigma_I} \, \omega^\sigma_I, \tag{8c}$$

$$d_V dx^i = 0\,, \qquad d_V dy^\sigma_I = dy^\sigma_{I+i} \wedge dx^i, \qquad d_V \omega^\sigma_I = 0. \tag{8d}$$

We note that $d_H dy^\sigma_I = d_H \omega^\sigma_I$.

1.10 Remark. A form $\alpha \in \Omega^n_r$ defines an *action functional*

$$\mathcal{A}(s, U) \overset{\text{def}}{=} \int_U (j_r s)^* \alpha, \tag{9}$$

where $U \subset X$ is any oriented n-dimensional submanifold of X with regular boundary. This is slightly more general than the usual notion, where a horizontal form of the type $\lambda\colon J^r\pi \to \wedge^n T^*X$ is used (see, *e.g.*, [37]). It follows that contact forms yield no contribution to action-like functionals. The definition (9) is a first motivation for the computations of the above section.

2 Finite order variational sequence

The first statement of a partial version of finite order variational sequence was in [3]. This finite order variational sequence stopped with a trivial projection to 0 just after the space of finite order source forms (see below). The local exactness of this sequence was proved, together with an original solution of the global inverse problem (despite the fact that in order to do that the authors used infinite order jets). For more detailed comments about that variational sequence see remark 2.8.

The first formulation of a (long) variational sequence on finite order jet spaces is due to Krupka [23] (see [25] for the case $n = 1$). Below we will describe the main points of the approach of [23], and compare it with other approaches.

In [23] a natural exact subsequence of the de Rham sequence on $J^r\pi$ is defined. This subsequence is made by contact forms and their differentials. Then we define the r–th order variational sequence to be the quotient of the de Rham sequence on $J^r\pi$ by means of the above exact subsequence. Local and global results about the variational sequence are proved using the fact that the above subsequence is globally exact and using the abstract de Rham theorem.

Let us consider the sheaf of 1-contact forms $\mathcal{C}^1\Omega^*_r$, and denote by $(d\mathcal{C}^p\Omega^k_r)\breve{}$ the sheaf generated by the presheaf $d\mathcal{C}^p\Omega^k_r$. We set

$$\begin{aligned} \Theta^q_r &\stackrel{\text{def}}{=} \mathcal{C}^1\Omega^q_r + (d\mathcal{C}^1\Omega^{q-1}_r)\breve{} \quad 0 \le q \le n, \\ \Theta^{p+n}_r &\stackrel{\text{def}}{=} \mathcal{C}^p\Omega^{p+n}_r + (d\mathcal{C}^p\Omega^{p+n-1}_r)\breve{} \quad 1 \le p \le \dim J^r\pi. \end{aligned} \tag{10}$$

We observe that $d\mathcal{C}^1\Omega^{q-1}_r \subset \mathcal{C}^1\Omega^q_r$, so that the second summand of the above first equation yields no contribution to $\mathcal{C}^1\Omega^q_r$. The sheaves Θ^{p+n}_r become trivial when $p+n > P$, where the value of P is computed in [23] using Theorem 1.3. Moreover, we have the following property (proved in [23]).

2.1 Lemma. *Let $0 \le k \le \dim J^r\pi$. Then the sheaves Θ^k_r are soft sheaves.*

We have the following natural soft subsequence of the de Rham sequence on $J^r\pi$

$$0 \longrightarrow \Theta^1_r \xrightarrow{d} \Theta^2_r \xrightarrow{d} \cdots \xrightarrow{d} \Theta^P_r \xrightarrow{d} 0 \tag{11}$$

2.2 Definition. The sheaf sequence (11) is said to be the *contact sequence.*

2.3 Theorem. *The contact sequence is an exact soft resolution of $\mathcal{C}^1\Omega^1_r$, hence the cohomology of the associated cochain complex of sections on any open subset of $J^r\pi$ vanishes.*

The above theorem is proved in [23] by first proving the local exactness of the contact sequence and then using standard results from sheaf theory (for which an adequate source is [50]).

Standard arguments of homological algebra prove that the following diagram is commutative, and its rows and columns are exact.

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & 0 & & 0 & & & & 0 & & & & \\
 & & & & & & \downarrow & & \downarrow & & & & \downarrow & & & & \\
 & & & & 0 & \longrightarrow & \Theta^1_r & \xrightarrow{d} & \Theta^2_r & \xrightarrow{d} & \dots & \xrightarrow{d} & \Theta^P_r & \xrightarrow{d} & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & & & \downarrow & & & & \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \Omega^0_r & \xrightarrow{d} & \Omega^1_r & \xrightarrow{d} & \Omega^2_r & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega^P_r & \xrightarrow{d} & \Omega^{P+1}_r & \xrightarrow{d} & \dots 0 \\
 & & & & & \searrow & \downarrow & & \downarrow & & & & \downarrow & \nearrow_{\mathcal{E}_I} & & & \\
 & & & & & & \Omega^1_r/\Theta^1_r & \xrightarrow{\mathcal{E}_1} & \Omega^2_r/\Theta^2_r & \xrightarrow{\mathcal{E}_2} & \dots & \xrightarrow{\mathcal{E}_{P-1}} & \Omega^P_r/\Theta^P_r & & & & \\
 & & & & & & \downarrow & & \downarrow & & & & \downarrow & & & & \\
 & & & & & & 0 & & 0 & & & & 0 & & & &
\end{array}
$$

2.4 Definition. The above diagram is said to be the *r-th order variational bicomplex* associated with the fibred manifold $\pi\colon Y \to X$. We say the bottom row of the above diagram to be the *r-th order variational sequence* associated with the fibred manifold $\pi\colon Y \to X$.

Due to theorem 2.3 the finite order variational sequence is an exact sheaf sequence (this means that the sequence is locally exact, [50]). Hence both the de Rham sequence and the variational sequence are acyclic resolutions of the constant sheaf $\mathbb{R}$ ('acyclic' means that the sequences are locally exact with the exception of the first sheaf $\mathbb{R}$). Next corollary follows by the abstract de Rham theorem.

2.5 Corollary. *The cohomology of the variational sequence is naturally isomorphic to the de Rham cohomology of $J^r\pi$.*

The above finite order diagram yields a variational sequence which can be proved to be equal to the finite order variational sequence obtained from a finite order analogue of the $\mathcal{C}$-spectral sequence [49]. Moreover, as one could expect, for $0 \leq s < r$ pull-back via $\pi_{r,s}$ yields a natural inclusion of the s-th order variational bicomplex into the r-th order variational bicomplex. More precisely, we have the following lemma (see [23]).

2.6 Lemma. *Let $0 \leq s < r$. Then we have the injective sheaf morphism*

$$\chi^r_s : \left(\Omega^k_s/\Theta^k_s\right) \to \left(\Omega^k_r/\Theta^k_r\right), \quad [\alpha] \mapsto [\pi^*_{r,s}\alpha].$$

Hence, there is an inclusion of the s–th order variational bicomplex into the r–th order variational bicomplex. The inclusion commutes with the operators of the variational bicomplexes of orders s and r.

Having already dealt with local and global properties of the r-th order variational sequence, we are left with the problem of representing the quotient sheaves. This problem has been independently solved by many authors in the infinite order case. We recognize two different approaches to the problem: with differential forms (see for example [41, 42]) and with differential operators [43, 44]. The restriction to finite order jets of the former approach has been developed in [46] for $p = 1$, $p = 2$, and in [20, 21] for all p. See [49] for a finite order differential operator approach. We will describe the differential forms approach.

First of all, it is obvious that, for $0 \leq q \leq n$, horizontalization provides such a representation (see [23, 46]).

2.7 Proposition. *Let $0 \leq q \leq n$. Then we have the isomorphism*

$$H_q \colon \Omega^q_r/\Theta^q_r \to \overline{\Omega}^{0,q}_r, \quad [\alpha] \mapsto h^{0,q}(\alpha).$$

The quotient differential $\mathcal{E}_q$ reads through the above isomorphism as

$$H_{q+1}(\mathcal{E}_q([\alpha])) = H_{q+1}([d\alpha]) = h^{0,q+1}(d\alpha) = d_H h^{0,q}(\alpha).$$

The last equality of the above equation is the least obvious, and was first proved in [3]. The proof depends on the fact that $D_i y^\sigma_{I+j} = y^\sigma_{I+j+i}$, and that the indexes i, j are skew-symmetrized in the coefficients of $d_H h^{0,q}(\alpha)$ (see the coordinate expression of $h^{0,q}$).

2.8 Remark. In [3] the finite order variational sequence is developed starting from the idea of finding a subsequence of forms whose order do not change under d_H. The authors prove that the above property characterizes the forms which are in the image of $h^{0,q}$ (see also [2]). Conversely, in [23] the idea is to start with forms on finite order jets, but the result is the same up to the degree $q = n$.

When the degree of forms is greater than n we are able to provide isomorphisms of the quotient sheaves with other quotient sheaves made with proper subsheaves. This helps both to the purpose of representing quotient sheaves and to the purpose of comparing the current approach with others, as we will see.

2.9 Proposition. *Let $p \geq 1$. The horizontalization $h^{p,n}$ induces the natural sheaf isomorphism*

$$H_{p+n} \colon \Omega^{p+n}_r/\Theta^{p+n}_r \to \overline{\Omega}^{p,n}_r / h^{p,n}((d\mathcal{C}^p\Omega^{p+n-1}_r)), \quad [\alpha] \mapsto [h^{p,n}(\alpha)].$$

The quotient differential $\mathcal{E}_{p+n}$ reads through the above isomorphism as

$$H_{p+1+n}(\mathcal{E}_{p+n}([\alpha])) = H_{p+1+n}([d\alpha]) = [h^{p+1,n}(d\alpha)].$$

For a proof, see [46, 48].

Following [2, 41, 42], let us introduce the map

$$I_p \colon \mathcal{C}^p\Omega^p_r \wedge \mathcal{H}^n_r \to \mathcal{C}^p\Omega^p_{2r} \wedge \mathcal{H}^n_{2r}, \quad I_p(\alpha) = \frac{1}{p}\omega^\sigma \wedge (-1)^{|I|} D_I \left(i_{\partial/\partial u^\sigma_I}\alpha\right) \qquad (12)$$

where D_I stands for the iterated Lie derivative $(L_{D_1})^{i_1} \cdots (L_{D_n})^{i_n}$. We say the map I_p to be the *interior Euler operator*. It can be proved [2, 20, 42] that the following properties of I_p holds

- I_p is a natural map, *i.e.*, $L_{X^{2r}}(I_p(\alpha)) = I_p(L_{X^r}(\alpha))$, hence I_p is a global map;
- if $\alpha \in \mathcal{C}^p\Omega^p_r \wedge \mathcal{H}^n_r$ then there exists a unique form $\beta \in \mathcal{C}^p\Omega^p_{2r} \wedge \mathcal{H}^n_{2r}$, which is of the type $\beta = d_H\gamma$ with $\gamma \in \mathcal{C}^p\Omega^p_{2r-1} \wedge \mathcal{H}^{n-1}_{2r-1}$, such that
$$\alpha = I(\alpha) + \beta. \tag{13}$$

2.10 Remark. The above form γ is not uniquely defined, in general. For $p = 1$, if the order of α is 1 it is easily proved that γ is uniquely defined; if the order of α is 2 then there exists a unique γ fulfilling a certain intrinsic property; if the order is 3 it is proved in [16, 17] that no natural γ of the above type exists. However, suitable linear connections on M and on the fibres of $\pi\colon E \to M$ can be used to determine a unique γ. See [1, 2] for the case of $p > 1$.

It follows from the above theorem that if $\gamma \in \mathcal{C}^p\Omega^p_{2r-1} \wedge \mathcal{H}^{n-1}_{2r-1}$ then $I_p(d_H\gamma) = 0$, so that $I_p^2 = I_p$.

2.11 Theorem. *We have the isomorphism*
$$\Omega^{p+n}_r / \Theta^{p+n}_r \to \mathcal{V}^p_r, \quad [\alpha] \mapsto I_p(H_{p+n}([\alpha])),$$
where $\mathcal{V}^p_r \subset \mathcal{C}^p\Omega^p_{2r+1} \wedge \mathcal{H}^n_{2r+1}$ *is a suitable subspace (see [46] for a characterization for* $p = 1$*,* $p = 2$*).*

For a proof, see [46] ($p = 1$, $p = 2$) and [20, 21] for any p. The above theorem also mean that, despite the fact that the denominator in proposition 2.9 is made by forms which are *locally* total divergences, only global divergences really matter. We say the elements of $\mathcal{V}^p$ to be the p-th degree *variational forms*; for $p = 1$ they are also known as *source forms*.

The map I_{p+1} allows us to represent the differentials $\mathcal{E}_{p+n}$ through forms:
$$I_{p+1}(\mathcal{E}_{p+n}([\alpha])) = I_{p+1}([d\alpha]). \tag{14}$$
From the coordinate expression of I_p it follows that $\mathcal{E}_n$ is just the Euler–Lagrange operator and $\mathcal{E}_{1+n}$ is just the Helmholtz operator. In fact, let $\nu \overset{\text{def}}{=} dx^1 \wedge \cdots \wedge dx^n$. Then, if $\lambda \in \overline{\Omega}^{0,n}_r$, then $\lambda = h^{0,n}(\alpha) = L\nu$, where L is a function with polynomial structure in $r + 1$-st order derivatives as in (5). Now we can use (14) on α, but if α is not known the computational problem of finding it can be technically difficult in principle. On the other hand, we can use the commutativity of the inclusion of Lemma 2.6 with the operators $\mathcal{E}_{p+n}$ and consider $\lambda \in \Omega^n_{r+1}$. Then $h^{0,n}(\lambda) = \lambda$ and $\mathcal{E}_n(\lambda)$ is the standard Euler–Lagrange operator on the $r+1$-st order Lagrangian λ. A similar reasoning proves that $\mathcal{E}_{1+n}$ coincide with the Helmholtz operator.

A different, computational approach to the problem of the representation of quotients is presented in [13, 14].

A further approach to the problem of representation appeared in [30] for the case $n = 1$. Here the concept of *Lepagean equivalent* is introduced in full generality (older version of this concept can be found *e.g.*, in [22], with references to older foundational works). Namely, let $\alpha \in \Omega^{p+n}_r$. Then a Lepage equivalent of $[\alpha] \in \Omega^{p+n}_r/\Theta^{p+n}_r$ is a differential form $\beta \in \Omega^{p+n}_r$ such that

$$h^{p,n}(\beta) = h^{p,n}(\alpha), \qquad h^{p+1,n}(d\beta) = I_{p+1}(h^{p+1,n}(d\alpha)).$$

The most important example of a Lepagean equivalent is the Poincaré–Cartan form of a Lagrangian (see, *e.g.*, [22]).

3 Some related problems

In this section we will briefly describe what are the most recent results which involve finite order variational sequences.

3.1 Variationally trivial Lagrangians.

A variationally trivial Lagrangian is an element $[\alpha] \in \Omega^n_r/\Theta^n_r$ such that $\mathcal{E}_n([\alpha]) = 0$. If $[\alpha]$ is a variationally trivial Lagrangian, then by the local exactness of the variational sequence we have $h^{0,n}(\alpha) = d_H(h^{0,n-1}(\beta))$ with $[\beta] \in \Omega^{n-1}_r/\Theta^{n-1}_r$ a local form. A global horizontal $n-1$-form $[\beta] \in \Omega^{n-1}_r/\Theta^{n-1}_r$ such that $[\alpha] = d_H[\beta]$ exists if and only if $[\alpha]$ induces the zero cohomology class in the variational sequence. A refinement of this result is the following theorem.

3.1 Theorem. *Let $\lambda\colon J^r\pi \to \wedge^n T^*X$ induce a variationally trivial Lagrangian $[\lambda]$. Then, locally, $\lambda = d_H\mu$, where $\mu = h^{0,n-1}(\alpha)$ and $\alpha \in \Omega^{n-1}_{r-1}$.*

In other words, $\lambda = h^{0,n}(d\alpha)$, hence λ is the representative of a class $\mathcal{E}_{n-1}([\alpha]) = [d\alpha] \in \Omega^{n-1}_{r-1}/\Theta^{r-1}_{n-1}$. This means that λ depends on r-th order derivatives through hyperjacobians. This result has been proved in [3], [4] (here the proof is for the special case when the Lagrangian does not depend on (x^i)), [14, 29] (here the proof uses the finite order variational sequence). See also [27] for another approach to the problem. Of course, the result is sharp: the order cannot be further lowered.

3.2 Locally variational source forms.

A locally variational source form is an element $[\alpha] \in \Omega^{n+1}_r/\Theta^{n+1}_r$ such that $\mathcal{E}_{1+n}([\alpha]) = 0$. If $[\alpha]$ is a locally variational source form, then by the local exactness of the variational sequence $[\alpha]$ is the Euler–Lagrange expression of a local Lagrangian, *i.e.*, $[\alpha] = \mathcal{E}_n([\beta])$ with $[\beta] \in \Omega^n_r/\Theta^n_r$. A global Lagrangian $[\beta] \in \Omega^n_r/\Theta^n_r$ such that $[\alpha] = \mathcal{E}_n([\beta])$ exists if and only if $[\alpha] = 0 \in H^{n+1}(Y)$.

The previous result is sharp with respect to the order [23, 46]. However, it can be very difficult to check that a source form is in the space $\Omega^{n+1}_r/\Theta^{n+1}_r$. A result proved in [2] is helpful in this sense. Let $y^{(r)}$ denote all derivative coordinates of order r on a jet space. Let $f \in C^\infty(J^{2r}\pi)$, and suppose that $f(x^i, y^{(0)}, \dots, y^{(r)}, ty^{(r+1)}, t^2y^{(r+2)}, \dots, t^ry^{(2r)})$ is a polynomial of degree less than or equal to r in $y^{(s)}$, with $r+1 \leq s \leq 2r$. Then f is said to be a weighted polynomial of degree r in the derivative coordinates of order $r+1 \leq s \leq 2r$.

3.2 Theorem. *Let $[\Delta]$ be a locally variational source form, with $\Delta\colon J^{2r}\pi \to C^*_0 \wedge \wedge^n T^*X$. Suppose that the coefficients of Δ are weighted polynomials of degree less than or equal to r. Then $\Delta = \mathcal{E}(\lambda)$, where $\lambda\colon J^r\pi \to \wedge^n T^*X$.*

Again, the result is sharp with respect to the order of the jet space where the Lagrangian is defined. The above theorem is complemented in [2] by a rather complex algorithm for building the lowest order Lagrangian. This algorithm is an improvement of the well-known Volterra Lagrangian

$$L = \int_0^1 y^\sigma \Delta_\sigma(x^i, ty^\tau_I)dt$$

for a locally variational source form Δ. In fact, the above Lagrangian is defined on the same jet space as Δ. The finite order variational sequence yields another method for computing lower order Lagrangians, provided we know that $\Delta = [\alpha] \in \Omega^{n+1}_r/\Theta^{n+1}_r$. Namely, we apply the contact homotopy operator to the closed form $d\alpha \in \Theta^{n+2}_r$, finding $\beta \in \Theta^{n+1}_r$ such that $d\beta = d\alpha$. Using the (standard) homotopy operator we find $\gamma \in \Omega^n_r$ such that $d\gamma = \beta - \alpha$, and $\lambda \stackrel{\text{def}}{=} h^{0,n}(\gamma)$ is the required Lagrangian. Of course, the most difficult point is to invert the representation of quotients in the variational sequence, *i.e.*, to find a least order α such that $\Delta = [\alpha]$.

The above theorem does not exhaust the finite order inverse problem. A locally variational source form Δ on $J^{2r}\pi$ seems to have a definite form of the coefficients with respect to its derivatives of order s, with $r+1 \leq s \leq 2r$. It is an open problem to determine such a structure, *e.g.* prove that such forms always lie in $\Omega^{n+1}_s/\Theta^{n+1}_s$ for a minimal value of s; a least order Lagrangian would follow from the local exactness of the variational sequence.

Finally, we recall that recently some geometric results on variational first-order partial differential equations have been obtained in [15]. Such equations arise in multisymplectic field theories.

3.3 Contact elements

Let Y be an $n+m$-dimensional manifold, and $x \in Y$. We say that two n-dimensional submanifolds L_1, L_2 such that $x \in L_1 \cap L_2$ are *r-equivalent* if they have a contact of order r at x. It is possible to choose a chart of Y at x of the form (x^i, y^σ), $1 \leq i \leq n$, $1 \leq i \leq m$, where both L_1 and L_2 can be expressed as graphs $y^\sigma = f^\sigma_1(x^i)$, $y^\sigma = f^\sigma_2(x^i)$. Then the contact condition is the equality of the derivatives of the above functions

f_1, f_2 at x up to the order r. This is an equivalence relation whose quotient set is $J^r(Y,n)$, the *manifold of r-th order n-dimensional contact elements*. This construction was first formalized in [10], and is also known as *r-th order jet space of n-dimensional submanifolds of* Y [45] or *extended jet bundle* [34]). If Y is endowed with a fibring π, then $J^r\pi$ is the open and dense subspace of $J^r(Y,n)$ which is made by submanifolds which are transverse to the fibring at a point (which, of course, can be locally identified with the images of sections, hence with local sections themselves).

Of course, manifolds of contact elements have a contact distribution, hence a variational sequence can be formulated through the $\mathcal{C}$-spectral sequence [7, 43, 44]. Manifolds of contact elements can also be seen as jets of parametrizations of submanifolds (*i.e.*, jets of local n-dimensional immersions) up to the action of the reparametrization group [18]. In this setting another approach to the variational sequence is [38]. In [33] the finite-order $\mathcal{C}$-spectral sequence on the manifold of contact elements is computed. Research based on Krupka's approach on a variational sequence on finite order contact elements is in progress [31].

Another interesting research topic is the development of finite order variational structures on differential equations, *i.e.* submanifolds of jet spaces. This would possibly lead to a classification of their conservation laws of a certain order [35].

3.4 Variational sequence and symmetries

The Lie derivative of variational forms is interesting for the determination of symmetries of Lagrangians and source forms. However, the result of a Lie derivative with respect to a prolonged vector field is a form which, in general, contains d_H-exact terms. For this reason it is natural to use a new operator, the variational Lie derivative, which is defined up to d_H-exact terms. Such a formula first appeared in [45] ('infinitesimal Stokes'formula') in the infinite order formalism. The finite order case has been dealt with in [11, 28]. See also [6] for symmetries of source forms which are locally but not globally variational. This topic has clear connections with Noether's theorem, for which we invite the reader to consult the above literature.

3.5 Further topics

We already mentioned that other approaches to variational sequences exist in literature, mostly on infinite order jets.

It can be proved that there exists an infinite order analogue of Krupka's r-th order variational bicomplex [47]. This is defined in view of Lemma 2.6 via a direct limit of the injective family of r-th order variational bicomplexes. Nonetheless the direct limit infinite order bicomplex will be a bicomplex of presheaves, because gluing forms defined on jets of increasing order provides 'forms' which are only locally of finite order.

The $\mathcal{C}$-spectral sequence on jets of fibrings yields an infinite order variational sequence [7, 43, 44]. See [26, 48] for a comparison with Krupka's approach and [49] for some finite order $\mathcal{C}$-spectral sequence computations.

In [36] the relationship between a part of the finite order variational sequence and the Spencer sequence are stressed. This relationship was already explored in [43, 44] in the case of infinite order jet spaces.

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