ON THE CORRESPONDENCE BETWEEN DIFFERENTIAL EQUATIONS AND SYMMETRY ALGEBRAS

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Abstract

Within the framework of inverse Lie problems we give some non–trivial examples of Lie remarkable equations, *i.e.*, classes of partial differential equations that are in correspondence with their Lie point symmetries.
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1 Introduction

Symmetries of differential equations (DEs), either ordinary or partial, are (finite or infinitesimal) transformations of the independent and dependent variables and derivatives of the latter with respect to the former, with the further property of sending solutions into solutions[2, 3, 4, 9, 15, 16].

The simplest symmetries one may consider are those coming from a transformation of the independent and dependent variables: *point symmetries*. The set of all infinitesimal point symmetries of a given DE has the structure of a Lie algebra.

The "direct" problem of finding the symmetries of a given DE can be complemented by a natural "inverse" problem, namely, the problem of finding the most general form of a DE admitting a given Lie algebra as subalgebra of its infinitesimal point symmetries. There are different ways to consider this inverse problem.

A first contribution[3] aimed at characterizing all DEs admitting a given symmetry group. In Ref. [17] the problem whether there exist non–trivial DEs in one–to–one correspondence with their Lie symmetries has been considered: the equation for the surfaces in $\mathbb{R}^3$ with vanishing Gaussian curvature has been recognized to be uniquely determined by its Lie point symmetries.

Many relevant DEs can not be characterized by their Lie point symmetries since the dimension of the admitted Lie algebra is too small: KdV equation, Burgers' equation, Kepler's equations, ... are just some examples. Some authors[17, 10] studied the problem of finding, for a given equation, an extension of the algebra of point symmetries for which the equation at hand can be determined. In Ref. [17] contact symmetries have been considered, whereas in Ref. [10], where the notion of *complete symmetry group* has been introduced to completely characterize the Kepler's equations, non–local symmetries have been used. The notion of complete symmetry group for second and third order ODEs has been also considered in Ref. [1].

The general approach to the inverse problem of constructing differential equations by using abstract Lie algebras requires to classify all possible realizations of the considered Lie algebra as algebra of vector fields on the space of independent and dependent variables. Then, looking for the differential invariants of the realization under consideration[8], under suitable hypotheses of regularity, the most general DE admitting a given Lie algebra as subalgebra of point symmetries is locally given as a zero set of smooth functions of the differential invariants.

In this paper we present an overview of recent results[12, 13] about the characterization of DEs by means of their Lie algebras of point symmetries (*Lie remarkable equations*). More in detail, we report some examples of DEs which are determined by their symmetries (Monge–Ampère equations, minimal surface equations) and we construct differential equations uniquely determined by some relevant Lie algebras of vector fields on $\mathbb{R}^3$.

In section 2, we introduce a DE of order r as a submanifold of a suitable jet space[4, 15] (of order r). Symmetries of a given DE will be interpreted as particular vector fields on the jet space tangent to the DE. Then we introduce two distinguished classes of Lie remarkable equations: *strongly* and *weakly* Lie remarkable equations. Strongly Lie remarkable equations are uniquely determined by their point symmetries in the whole jet space; weakly Lie remarkable equations are equations which do not intersect other equations admitting the same symmetries.

We find necessary and sufficient conditions for a given DE to be (strongly or weakly) Lie remarkable by analyzing the dimension of the Lie algebra of point symmetries and the regularity of the local action that these symmetries induce on the jet space where the DE is immersed. Our viewpoint reverses and generalizes the Lie determinant method [16].

In the remaining sections we give various examples of DEs that are strongly or weakly Lie remarkable. Such examples are derived in two ways. In section 3 we consider the

Lie algebra of point symmetries of a given DE and prove that the equation is (strongly or weakly) Lie remarkable with respect to its symmetry group. We obtain two main classes of examples: Monge–Ampère equations and minimal submanifold equations. In section 4, we find strongly Lie remarkable equations associated with isometric, affine, projective and conformal algebra of $\mathbb{R}^3$, with respect to metrics of various signatures. Since we start from concrete algebras and not from abstract ones, we do not have the problem of realizing them as vector fields. In particular, with regard to the affine algebra in $\mathbb{R}^3$ we recover the homogeneous second order Monge-Ampère equation [5], and a third order PDE that, to the author's knowledge has not been described heretofore in literature. Also in the case of conformal algebra in $\mathbb{R}^3$, an interesting second order PDE is recovered, *i.e.* the equation for a surface $u(x, y)$ having the square of the (scalar) mean curvature equal to the Gaussian curvature.

2 Theoretical setting

Here we recall some basic facts regarding Lie remarkable equations[12, 13]. The theory is carried out in the geometric framework of jet bundles (for more details, see Refs. [4, 15]).

All manifolds and maps are supposed to be C^∞. If E is a manifold then we denote by $\chi(E)$ the Lie algebra of vector fields on E.

Let E be an $(n+m)$-dimensional smooth manifold. We make use of local charts of the form (x^λ, u^i), $\lambda = 1 \dots n$ and $i = 1 \dots m$, and we describe (locally) an n-dimensional submanifold $L \subset E$ as the graph of a vector function $u^i = f^i(x^\lambda)$. In what follows, Greek indices run from 1 to n and Latin indices run from 1 to m unless otherwise specified.

The *r-jet of n-dimensional submanifolds of E* (also known as extended jet bundles [15], or manifold of contact elements), $J^r(E, n)$, is the set of equivalence classes of submanifolds having at $p \in E$ a contact of order r. It has a smooth manifold structure: the charts are $(x^\lambda, u^i_{\boldsymbol{\sigma}})$, where $u^i_{\boldsymbol{\sigma}} \circ j_r L = \partial^{|\boldsymbol{\sigma}|} f^i / \partial x^{\boldsymbol{\sigma}}$, and $\boldsymbol{\sigma}$ is a multiindex such that $0 \leq |\boldsymbol{\sigma}| \leq r$. Hence we have $\dim J^r(E, n) = n + m\binom{n+r}{r}$. On $J^r(E, n)$ there is a distribution, the contact distribution, which is generated by the total derivatives $D_\lambda \overset{\text{def}}{=} \partial/\partial x^\lambda + u^j_{\boldsymbol{\sigma}\lambda} \partial/\partial u^j_{\boldsymbol{\sigma}}$ and $\partial/\partial u^j_{\boldsymbol{\tau}}$, where $0 \leq |\boldsymbol{\sigma}| \leq r-1$, $|\boldsymbol{\tau}| = r$ and $\boldsymbol{\sigma}\lambda$ denotes the multi-index $(\sigma_1, \dots, \sigma_{r-1}, \lambda)$. Any vector field $X \in \chi(E)$ can be lifted to a vector field $X^{(r)} \in \chi(J^r(E, n))$ which preserves the contact distribution. In coordinates, if $X = X^\lambda \partial/\partial x^\lambda + X^{n+i} \partial/\partial u^i$, then we have the well known formula $X^{(k)} = X^\lambda \partial/\partial x^\lambda + X^{n+i}_{\boldsymbol{\sigma}} \partial/\partial u^i_{\boldsymbol{\sigma}}$, where $X^{n+i}_{\boldsymbol{\tau},\lambda} = D_\lambda(X^{n+i}_{\boldsymbol{\tau}}) - u^i_{\boldsymbol{\tau},\beta} D_\lambda(X^\beta)$ with $|\boldsymbol{\tau}| < k$.

A *differential equation $\mathcal{E}$ of order r on n-dimensional submanifolds of a manifold E* is a submanifold of $J^r(E, n)$. An *infinitesimal point symmetry* of $\mathcal{E}$ is a vector field of the type $X^{(r)}$ which is tangent to $\mathcal{E}$. If $\mathcal{E}$ is locally described by $\{F^i = 0\}$, $i = 1 \dots k$ with $k < \dim J^r(E, n)$, then point symmetries are the solutions of the system $X^{(r)}(F^i) = 0$ whenever $F^i = 0$. We denote by $\mathrm{sym}(\mathcal{E})$ the Lie algebra of infinitesimal point symmetries of the equation $\mathcal{E}$.

Now we summarize the definitions and the main properties, contained in Ref. [12], of DEs which are characterized by their point symmetries, that we call *Lie remarkable.*

1 Definition. Let E be a manifold, $\dim E = n + m$, and let $r \in \mathbb{N}$, $r > 0$. An l-dimensional equation $\mathcal{E} \subset J^r(E, n)$ is said to be

1. *weakly Lie remarkable* if $\mathcal{E}$ is the only maximal (with respect to the inclusion) l-dimensional equation in $J^r(E, n)$ passing at any $\theta \in \mathcal{E}$ and admitting $\mathrm{sym}(\mathcal{E})$ as subalgebra of the algebra of its infinitesimal point symmetries;

2. *strongly Lie remarkable* if $\mathcal{E}$ is the only maximal (with respect to the inclusion) l-dimensional equation in $J^r(E, n)$ admitting $\mathrm{sym}(\mathcal{E})$ as subalgebra of the algebra of its infinitesimal point symmetries.

Of course, a strongly Lie remarkable equation is also weakly Lie remarkable. Some direct consequences of our definitions are in order. For each $\theta \in J^r(E, n)$, let us denote by $S_\theta(\mathcal{E}) \subset T_\theta J^r(E, n)$ the subspace generated by the values of infinitesimal point symmetries of $\mathcal{E}$ at θ. Let us set $S(\mathcal{E}) \stackrel{\mathrm{def}}{=} \bigcup_{\theta \in J^r(E,n)} S_\theta(\mathcal{E})$. In general, $\dim S_\theta(\mathcal{E})$ may change with $\theta \in J^r(E, n)$. It is clear that $\dim \mathrm{sym}(\mathcal{E}) \geq S_\theta(\mathcal{E})$, for all $\theta \in J^r(E, n)$. If the rank of $S(\mathcal{E})$ at each $\theta \in J^r(E, n)$ equals $\dim \mathrm{sym}(\mathcal{E})$, then $S(\mathcal{E})$ is an involutive (smooth) distribution. The points of $J^r(E, n)$ of maximal rank of $S(\mathcal{E})$ form an open set of $J^r(E, n)$ (Ref. [12]). It follows that $\mathcal{E}$ can not coincide with the set of points of maximal rank of $S(\mathcal{E})$. The following statements (see Ref. [12]) can be proved.

1. A necessary condition for the differential equation $\mathcal{E}$ to be strongly Lie remarkable is that $\dim \mathrm{sym}(\mathcal{E}) > \dim \mathcal{E}$.

2. A necessary condition for the differential equation $\mathcal{E}$ to be weakly Lie remarkable is that $\dim \mathrm{sym}(\mathcal{E}) \geq \dim \mathcal{E}$.

3. If $S(\mathcal{E})|_{\mathcal{E}}$ is an l-dimensional distribution on $\mathcal{E} \subset J^r(E, n)$, then $\mathcal{E}$ is a weakly Lie remarkable equation.

4. Let $S(\mathcal{E})$ be such that for any $\theta \notin \mathcal{E}$ we have $\dim S_\theta(\mathcal{E}) > l$. Then $\mathcal{E}$ is a strongly Lie remarkable equation.

Several examples of strongly and weakly Lie remarkable equations are provided in next sections. In the sequel, to make notation lighter, when $n = 2$, we will use x and y instead of x^1 and x^2, respectively.

3 DEs characterized by their Lie point symmetries

Minimal surfaces in $\mathbb{R}^{n+m}$. Let $E = \mathbb{R}^{n+m}$ endowed with the standard Euclidean metric. In the case $n = 2$, $m = 1$, the mean and Gaussian curvatures are the real functions on $J^2(\mathbb{R}^3, 2)$ defined by

$$H = \frac{1}{2}\frac{(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy}}{(1 + u_x^2 + u_y^2)^{3/2}}, \quad G = \frac{u_{xx}u_{yy} - u_{xy}^2}{(1 + u_x^2 + u_y^2)^2}.$$

The mean curvature can be generalized to surfaces in $\mathbb{R}^{2+m}$, with $m \geq 2$. The minimal surface equation $\mathcal{E}$ is then the equation $\{H = 0\}$. A computation shows that the point symmetries of $\{H = 0\}$ are the isometries and the homotheties of $\mathbb{R}^{2+m}$ for $1 \leq m \leq 4$. A more sophisticated computation proves that $S(\mathcal{E})$ has maximal rank on an open subset of $\mathcal{E}$.

If d is the dimension of the minimal surface equation in $\mathbb{R}^{2+m}$ and i is the dimension of the group made by isometries of $\mathbb{R}^{2+m}$ and homotheties, then it turns out that, for $m = 2$ and $m = 3$ we have $d > i$ while for $m = 1$ and $m = 4$ we have $d = i$.

The above two arguments, together with the necessary and sufficient conditions of the previous section, allow us to prove the following theorem.

2 Theorem. *The equation of minimal surfaces in $\mathbb{R}^4$ and $\mathbb{R}^5$ is neither strongly nor weakly Lie remarkable, whereas it is weakly Lie remarkable in $\mathbb{R}^3$ and $\mathbb{R}^6$, provided that we remove a singular equation.*

Monge-Ampère equations. We start with the following Monge–Ampère equation

$$u_{xx}u_{yy} - u_{xy}^2 = \kappa, \tag{1}$$

which, in the case $\kappa = 0$, is just the equation $G = 0$ for surfaces. If $\kappa \neq 0$, then Eq. (1) admits a 9-parameter group of point symmetries which span a 7-dimensional distribution on the jet space except for a singular lower-dimensional submanifold. On the other hand, if $\kappa = 0$ then Eq. (1) admits a 15–parameter group of point symmetries whose associated distribution is 8-dimensional (provided that we remove a singular lower–dimensional submanifold). Hence the following result arises.

3 Theorem. *Eq.* (1) *is weakly Lie remarkable if $\kappa \neq 0$, whereas it is strongly Lie remarkable if $\kappa = 0$.*

As shown by Boillat [5], Eq. (1) has the property of complete exceptionality. The use of such a property permitted to Boillat[6] to introduce higher order equations that are called generalized Monge–Ampère equations. Among them one can find several Lie remarkable equations[11, 12]. Just as an example, the third order Monge–Ampère equation

$$(u_{xxy}u_{yyy} - u_{xyy}^2) + \lambda(u_{xxx}u_{yyy} - u_{xxy}u_{xyy}) + \lambda^2(u_{xxx}u_{xyy} - u_{xxy}^2) + \mu = 0,$$

where λ and μ are constants, is weakly Lie remarkable.

4 DEs which are uniquely determined by Lie algebras of vector fields on $\mathbb{R}^{n+m}$

In what follows we shall consider only scalar partial differential equations in two independent variables, *i.e.*, $n = 2$, $m = 1$, $E = \mathbb{R}^3$. We denote by $\mathcal{I}(\mathbb{R}^3)$, $\mathcal{A}(\mathbb{R}^3)$, $\mathcal{P}(\mathbb{R}^3)$ and

$\mathcal{C}(\mathbb{R}^3)$, respectively, the isometric, affine, projective and conformal algebras of $\mathbb{R}^3$ with respect to the metric $g = k_1 dx \otimes dx + k_2 dy \otimes dy + du \otimes du$, where k_1, k_2 are non-vanishing real constants.

The problem that we will recall here (see Ref. [13]) consists in finding the strongly Lie remarkable equations associated with the previous Lie algebras of infinitesimal transformations.

The case $\mathcal{I}(\mathbb{R}^3)$. The algebra $\mathcal{I}(\mathbb{R}^3)$ has dimension 6. Then, in view of the necessary conditions, there can be strongly Lie remarkable equations only of order 1. In order to find them one prolongs the vector fields to the first jet space and computes the rank of the generated distribution. Such a rank decreases only on the singular submanifold

$$1 + \frac{u_x^2}{k_1} + \frac{u_y^2}{k_2} = 0, \tag{2}$$

which turns out to be a strongly Lie remarkable equation. Of course, the equation is nonempty if and only if the constants k_i are not all positive.

The case $\mathcal{A}(\mathbb{R}^3)$. The algebra $\mathcal{A}(\mathbb{R}^3)$ has dimension 12. By using the necessary conditions, we see that there can be strongly Lie remarkable equations of order 2 or 3.

A computation[13] shows that the strongly Lie remarkable second order equation is the homogeneous Monge–Ampère equation $G = 0$, and that there exists also a strongly Lie remarkable equation of third order which has the following local expression:

$$\begin{aligned} &u_{xx}^3 u_{yyy}^2 + u_{xxx}^2 u_{yy}^3 + 6u_{xx}u_{xxx}u_{xy}u_{yy}u_{yyy} - 6u_{xxx}u_{xxy}u_{xy}u_{yy}^2 \\ &-6u_{xx}u_{xxx}u_{xyy}u_{yy}^2 - 6u_{xx}^2 u_{xy}u_{xyy}u_{yyy} - 6u_{xx}^2 u_{xxy}u_{yy}u_{yyy} \\ &-8u_{xxx}u_{xy}^3 u_{yyy} + 9u_{xx}u_{xxy}^2 u_{yy}^2 + 9u_{xx}^2 u_{xyy}^2 u_{yy} \\ &+12u_{xxx}u_{xy}^2 u_{xyy}u_{yy} + 12u_{xx}u_{xxy}u_{xy}^2 u_{yyy} - 18u_{xx}u_{xxy}u_{xy}u_{xyy}u_{yy} = 0. \end{aligned} \tag{3}$$

To the authors' knowledge, equation (3) has not been heretofore described in literature.

The case $\mathcal{P}(\mathbb{R}^3)$. The algebra $\mathcal{P}(\mathbb{R}^3)$ has dimension 15. Then, as in the previous case, equations $G = 0$ and (3) are strongly Lie remarkable.

The case $\mathcal{C}(\mathbb{R}^3)$. The algebra $\mathcal{C}(\mathbb{R}^3)$ has dimension equal to 10. We have to look for second order strongly Lie remarkable equations. By analyzing the rank of the matrix of 2-prolongations of the vector fields we realize that the unique second order equation which is strongly Lie remarkable with respect to the conformal algebra is $G = H^2$. By a direct computation, we realize that the unique second order scalar differential invariant I of the algebra formed by $\mathcal{I}(\mathbb{R}^3)$ with the addition of homotheties[1] is $I = H^2/G$.

[1]We recall that a differential invariant is a function on a jet space which is invariant under the prolonged action of the vector fields of the given Lie algebra.

Then $I = k$, with k constant, is a weakly Lie remarkable equation. Therefore we could look for strongly Lie remarkable equations among the equations $I = k$. From the above discussion, it follows that $I = 1$ is the strongly Lie remarkable equation we were looking for.

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