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Quantum structures in Einstein general relativity

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Abstract

We introduce the notion of a "quantum structure" on an Einstein general relativistic classical spacetime $\boldsymbol{M}$. It consists of a line bundle over $\boldsymbol{M}$ equipped with a connection fulfilling certain conditions. We give a necessary and sufficient condition for the existence of quantum structures, and classify them. The existence and classification results are analogous to those of geometric quantisation (Kostant and Souriau), but they involve the topology of spacetime, rather than the topology of the configuration space. We provide physically relevant examples, such as the Dirac monopole, the Aharonov–Bohm effect and the Kerr–Newman spacetime.

Our formulation is carried out by analogy with the geometric approach to quantum mechanics on a spacetime with absolute time, given by Jadczyk and Modugno.

Key words: Einstein general relativity, particle mechanics, jets, non-linear connections, cosymplectic forms, geometric quantisation, cohomology.

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Introduction

With the view of the unification of quantum theories and general relativistic theories of gravitation an important achievement would be a covariant formulation of quantum mechanics, which requires a geometric approach.

Important non relativistic geometric formulations of quantum mechanics have been proposed so far: we mention the geometric quantisation [8, 19, 28, 29, 37] and the deformation quantisation [2].Several authors have proposed covariant formulations of classical and quantum mechanics on a curved spacetime with absolute time (see, for example, [4, 7, 20, 21, 30, 31]).

Recently, a covariant formulation has been presented of Galilei classical mechanics and quantum mechanics on a curved spacetime with absolute time and spacelike Riemannian metric, based on jets, connections and cosymplectic forms (see [12, 13] for the scalar case, and [3] for a generalisation to spin). This formulation presents some analogies and several novelties with respect to geometric quantisation. Namely, it is manifestly covariant with respect to any frame; accordingly, time is not just a parameter, but it plays a deep role in the geometric structure of the theory. In the flat case, this theory reduces to the standard quantum mechanics, hence recovers all standard examples (i.e. , hydrogen atom and harmonic oscillator). New non–standard examples have been studied as well [35]. In this context, the problem of existence and classification of inequivalent quantum structures has been solved [25, 33, 34].

The above theory stands in between the standard non relativistic quantum theory and a possibile general relativistic theory on a Lorentz manifold. Then, it would be interesting to investigate an Einstein version of the Galilei theory. Indeed, several procedures and results of the Galilei case have been successfully transferred to the Einstein case. In particular, [16] formulates the classical phase space in Einstein's general relativity in terms of jets and equips it with a cosymplectic form (in the sense of [1, 6, 22]) which incorporates the gravitational and electromagnetic fields. Moreover, [17] introduces a Lie algebra of quantisable functions and [18] proves that it is isomorphic to a Lie algebra of pre–quantum operators. Furthermore, [15] derives the Klein-Gordon equation by covariance arguments. We are aware of the well known difficulties for physical interpretation of the probability current and for the formulation of the Hilbert stuff. Perhaps, the solution of these problems might arise from a non linear generalisation of the standard quantum structures reflecting the non linearity of the underlying spacetime manifold.

In this paper, we formulate a definition of Einstein's general relativistic quantum structure in analogy with the Galilei case. Namely, we define a quantum structure to be a Hermitian complex line bundle over spacetime endowed with a "universal" connection whose curvature is proportional to the cosymplectic form.

Moreover, we prove a theorem of Kostant–Souriau type (see, for instance, [19, 29]), which states a necessary and sufficient condition for existence of quantum structures involving the topology of spacetime and the cosymplectic form. Also, we classify quantum structures by means of a topological invariant of the spacetime manifold. Finally, we illustrate the above formulation and results by means of some physically relevant examples. In particular, we consider the cases of Minkowski spacetime, Schwartzschild spacetime, Dirac monopole, Aharonov–Bohm effect and Kerr–Newman spacetime.

We end this introduction by some mathematical preliminaries.

The theory of *unit space* has been developed in [12, 13] in order to make explicit the independence of classical and quantum mechanics from the choice of unit of measurements. Unit spaces have the same algebraic structure as $I\!R_+$, but no natural basis. We assume the (one–dimensional) unit spaces $\mathbb{T}$ (space of *time intervals*), $\mathbb{L}$ (space of *lengths*) and $\mathbb{M}$ (space of *masses*). We set $\mathbb{T}^{-1} \equiv \mathbb{T}^*$, and analogously for $\mathbb{L}, \mathbb{M}$.

We assume the following constant elements: the *light velocity* $c \in \mathbb{T}^{-1} \otimes \mathbb{L}$, the *Planck's constant* $\hbar \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$, and the *gravitational coupling constant* $\kappa \in \mathbb{T}^{-2} \otimes \mathbb{L}^3 \otimes \mathbb{M}^{-1}$. Moreover, we say a *charge* to be an element $q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2} \otimes I\!R$.

We will assume coordinates to be dimensionless (i.e. , real valued). We assume manifolds and maps to be C^∞.

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1 Classical structures

In this section we summarise the results on the geometry of the phase space given by Janyška and Modugno [16]. By the way, we see that the geometric constructions of Galilei general relativistic spacetime [12, 13] can be recovered in Einstein's case. The major difficulty is that many objects which live on spacetime in Galilei's case, are defined on the phase space in Einstein's case.

1.1 Spacetime and phase space

Assumption C.1 We assume the *spacetime* to be a manifold $\boldsymbol{M}$, with $\dim \boldsymbol{M} = 4$, endowed with a scaled Lorentz metric $g : \boldsymbol{M} \to \mathbb{L}^2 \otimes T^*\boldsymbol{M} \underset{\boldsymbol{M}}{\otimes} T^*\boldsymbol{M}$ whose signature is $(+ - - -)$. Moreover, we assume $\boldsymbol{M}$ to be oriented and time–like oriented. □

Charts on $\boldsymbol{M}$ are denoted by (x^φ), $\varphi = 0, 1, 2, 3$. An element $u_0 \in \mathbb{T}$, or, equivalently, its dual $u^0 \in \mathbb{T}^{-1}$, is said to be a *time unit of measurement*, and will be used in coordinate expressions throughout. We have the coordinate expressions $g = g_{\varphi\psi} d^\varphi \otimes d^\psi$, where $g_{\varphi\psi} : \boldsymbol{M} \to \mathbb{L}^2 \otimes I\!R$.

In what follows we will use charts such that ∂_0 is time–like and time–like oriented, and $\partial_1, \partial_2, \partial_3$ are space–like; hence $g_{00} > 0$, $g_{11}, g_{22}, g_{33} < 0$. Latin indexes $i, j, p, \dots$ will label space–like coordinates, greek indexes $\lambda, \mu, \varphi, \dots$ will label spacetime coordinates.

A one–jet of a one–dimensional submanifold $s \subset \boldsymbol{M}$ at $x \in \boldsymbol{M}$ is defined to be the equivalence class of one–dimensional submanifolds having a contact with s of order one at x [24]. The equivalence class is denoted by $j_1 s(x)$, and the quotient set by $J_1(\boldsymbol{M}, 1)$. The set $J_1(\boldsymbol{M}, 1)$ has a natural manifold structure and a natural bundle structure $\pi^1_0 : J_1(\boldsymbol{M}, 1) \to \boldsymbol{M}$. A time–like one–dimensional submanifold $s \subset \boldsymbol{M}$ is said to be a *motion*, whose *velocity* is $j_1 s$. The set $U_1\boldsymbol{M}$, of velocities of motions is said to be the *phase space*. By a restriction we have the natural bundle structure $\pi^1_0 : U_1\boldsymbol{M} \to \boldsymbol{M}$. A section $o : \boldsymbol{M} \to U_1\boldsymbol{M}$ is said to be an *observer*. A typical chart (x^0, x^i) on $\boldsymbol{M}$ induces a local fibred chart $(x^0, x^i; x^i_0)$ on $U_1\boldsymbol{M}$. More precisely, if $s \subset \boldsymbol{M}$ is a one–dimensional submanifold such that $x^i|_s = s^i \circ x^0|_s$, then $x^i_0 \circ j_1 s = \partial_0 s^i = (Ds^i) \circ (x^0|_s)$.

The fibration π^1_0 induces the *contact structure* on $U_1\boldsymbol{M}$ [24] (which is the analogue of the contact structure on jet spaces of fibred manifolds)

$$\text{д}_1 : U_1\boldsymbol{M} \to \mathbb{T}^* \otimes T\boldsymbol{M}\,, \qquad \tau := c^{-2}\, g^\flat \circ \text{д}_1 : U_1\boldsymbol{M} \to \mathbb{T} \otimes T^*\boldsymbol{M}\,,$$

with coordinate expressions

$$\text{д}_1 = c\alpha\, \text{д}_{10} = c\alpha\, (\partial_0 + x^i_0 \partial_i)\,, \qquad \tau \equiv \tau_\lambda\, d^\lambda = c\alpha u_0\, (g_{0\lambda} + g_{i\lambda} x^i_0) d^\lambda\,.$$

where $\alpha = 1/\|\text{д}_{10}\|_g = 1/\sqrt{g_{00} + 2g_{0j}x^j_0 + g_{ij}x^i_0 x^j_0} \in \mathbb{L}^{-1}$.

We have $g \circ (\text{д}_1, \text{д}_1) = c^2$. Hence $U_1\boldsymbol{M}$ can be regarded as a non–linear subbundle $U_1\boldsymbol{M} \subset \mathbb{T}^* \otimes T\boldsymbol{M}$, whose fibres are diffeomorphic to $I\!R^3$.

If s is a motion, then $\text{д}_1 \circ js : s \to \mathbb{T}^* \otimes T\boldsymbol{M}$ is the vector field representing the velocity of s.

The metric g yields an orthogonal splitting of the tangent space $T\boldsymbol{M}$ on each $x \in \boldsymbol{M}$ on which a time–like direction has been assigned. In other words, we have the splitting [16] $U_1\boldsymbol{M} \underset{M}{\times} T\boldsymbol{M} = T^\parallel \boldsymbol{M} \underset{U_1\boldsymbol{M}}{\oplus} T^\perp \boldsymbol{M}$. The projection on $T^\perp\boldsymbol{M}$ is denoted by θ, with coordinate expression $\theta = h^{i\mu} h_{i\nu} d^\nu \otimes \partial_\mu$, where we have set $h_{i\nu} := g_{i\nu} - c^2 \tau_i \tau_\nu$ and $h^{i\mu} := g^{i\mu} - x^i_0 g^{0\mu}$.

The vertical derivative $V\text{д}_1$ induces the linear fibred isomorphism $v^\perp : VU_1\boldsymbol{M} \to \mathbb{T}^* \otimes T^\perp\boldsymbol{M}$ over $U_1\boldsymbol{M}$, with coordinate expression $v^\perp = c\alpha\, d^i_0 \otimes (\partial_i - c\alpha\tau_i \text{д}_{10})$.

1.2 Gravitational and electromagnetic forms

The Levi–Civita connection $K^\natural$ on $T\boldsymbol{M} \to \boldsymbol{M}$ induces naturally a (non linear) connection $\Gamma^\natural$ on $U_1\boldsymbol{M} \to \boldsymbol{M}$ [16], which is expressed by a section $\Gamma : U_1\boldsymbol{M} \to T^*\boldsymbol{M} \underset{U_1\boldsymbol{M}}{\otimes} TU_1\boldsymbol{M}$, and has the coordinate expression

$$\Gamma = d^\varphi \otimes (\partial_\varphi + \Gamma_\varphi{}^i{}_0 \partial^0_i)\,, \tag{1}$$

with $\Gamma_{\varphi 0}^{\;i} = K_{\varphi}{}^{i}{}_{j} x_0^j + K_{\varphi}{}^{i}{}_{0} - x_0^i (K_{\varphi}{}^{0}{}_{j} x_0^j + K_{\varphi}{}^{0}{}_{0})$. The connections $K^\natural$ and $\Gamma^\natural$ are said to be *gravitational*.

Let m be a mass. Then, the gravitational connection $\Gamma^\natural$ and the metric g induce the form on $U_1\boldsymbol{M}$

$$\Omega^\natural := \frac{m}{\hbar}\,(v^\perp \circ \nu_{\Gamma^\natural}) \overline{\wedge} \theta : U_1\boldsymbol{M} \to \overset{2}{\wedge} T^* U_1\boldsymbol{M}\,,$$

where $\overline{\wedge}$ denotes $\wedge$ followed by a contraction with g, and the factor $m/\hbar$ is put in order to obtain a non scaled object.

Definition 1.1 The above cosymplectic (in the sense of [1, 6, 22]) form $\Omega^\natural$ is said to be the *gravitational form*. □

Remark 1.1 Let us set $\tau^\natural := (mc^2)/\hbar\,\tau : U_1\boldsymbol{M} \to T^*\boldsymbol{M}$. Then, we can prove that the form $\Omega^\natural$ is an exact form on $U_1\boldsymbol{M}$. Moreover, the form $\tau^\natural$ is a distinguished potential of $\Omega^\natural$ [16].

Finally, $\Omega^\natural$ is non degenerate in the sense that

$$\tau^\natural \wedge \Omega^\natural \wedge \Omega^\natural \wedge \Omega^\natural$$

is a volume form on $U_1\boldsymbol{M}$. □

We have the coordinate expression

$$\Omega^\natural = \frac{m}{\hbar}\,\alpha h_{i\mu}(d_0^i - \Gamma_{\varphi 0}^{\;i}) \wedge d^\mu\,.$$

Now, we introduce the electromagnetic field.

Assumption C.2 We assume the *electromagnetic field* to be a closed form on $\boldsymbol{M}$

$$f : \boldsymbol{M} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}) \otimes \wedge^2 T^*\boldsymbol{M}\,. \;\square$$

Given a charge q, it is convenient to introduce the "normalised" electromagnetic field $F \equiv q/(\hbar c)\, f : \boldsymbol{M} \to \wedge^2 T^*\boldsymbol{M}$. We denote a local potential of F with $A : \boldsymbol{M} \to T^*\boldsymbol{M}$, according to $2dA = F$.

The forms on $U_1\boldsymbol{M}$ $E := -д \lrcorner F$ and $B := F + 2\tau^\natural \wedge E$, with

$$E : U_1\boldsymbol{M} \to \mathbb{T}^* \otimes T_\perp^*\boldsymbol{M}\,, \quad B : U_1\boldsymbol{M} \to \wedge^2 T_\perp^*\boldsymbol{M}\,,$$

are said to be, respectively, the *universal electric field* and the *universal magnetic field.* We have $F = -2\tau^\natural \wedge E + B$. We can read E and B through an observer $o : \boldsymbol{M} \to U_1\boldsymbol{M}$; in a chart adapted to o, i.e. a chart such that $o_0^i = 0$, we have

$$o^*E = -\frac{1}{\sqrt{g_{00}}} F_{0j} u^0 \otimes d^j\,, \quad o^*B = (F_{ij} - \frac{1}{g_{00}}(g_{i0}F_{0j} - g_{j0}F_{0i}))d^i \wedge d^j\,. \;\square$$

The electromagnetic field F can be incorporated into the geometrical structure of the phase space, i.e. the gravitational form. Namely, we define the *total form*

$$\Omega := \Omega^\natural + \frac{1}{2} F : U_1\boldsymbol{M} \to \wedge^2 T^* U_1 \boldsymbol{M} \,.$$

It is clear that Ω is a closed form, i.e. $d\Omega = 0$; anyway, Ω does not have, in general, a global potential. Locally, we can write

$$(\tau^\natural + A) \wedge \Omega \wedge \Omega \wedge \Omega = (\tau^\natural + A) \wedge \Omega^\natural \wedge \Omega^\natural \wedge \Omega^\natural \,,$$

so Ω is non degenerate. Hence, Ω is a cosymplectic form encoding gravitational and electromagnetic (classical) structures. By the way, we recall that a unique connection Γ on $U_1\boldsymbol{M} \to \boldsymbol{M}$ can be characterised through Ω [16].

2 Quantum bundle and quantum connection

A covariant formulation of the quantisation of classical mechanics of one particle can be developed in Einstein general relativity by analogy with the Galilei general relativistic case [12, 13]. One of the major difficulties occurs in the geometrical structure of the phase space, which is more complicated in Einstein's case than in Galilei's case.

We make use of the theory presented in the previous section in order to develop the geometric structures for the quantisation of the mechanics of one particle in an Einstein general relativistic background. [32, 33]. In particular, we introduce the *quantum bundle* and the *quantum connection*.

We refer to a particle with mass m and charge q.

Definition 2.1 A *quantum bundle* is defined to be a complex line bundle $\boldsymbol{Q} \to \boldsymbol{M}$ on spacetime, endowed with a Hermitian metric h. □

Two complex line bundles $\boldsymbol{Q}$, $\boldsymbol{Q}'$ on $\boldsymbol{M}$ are said to be *equivalent* if there exists an isomorphism of complex line bundles $f : \boldsymbol{Q} \to \boldsymbol{Q}'$ on $\boldsymbol{M}$. In this case, if $\boldsymbol{Q}$, $\boldsymbol{Q}'$ are Hermitian, then $\boldsymbol{Q}$, $\boldsymbol{Q}'$ are also isometric. The set of equivalence classes of (Hermitian) complex line bundles $\mathcal{L}(\boldsymbol{M})$ is isomorphic to $H^2(\boldsymbol{M}, \mathbb{Z})$ [8, 36].

Quantum histories are represented by *quantum sections* $\Psi : \boldsymbol{M} \to \boldsymbol{Q}$. A normalised complex adapted chart on $\boldsymbol{Q}$ is denoted by (x^0, x^i, z), and the corresponding local base for quantum sections is denoted by b. Hence, a quantum section has the coordinate expression $\Psi = \psi b$.

We denote by $\text{и} : \boldsymbol{Q} \to V\boldsymbol{Q} \simeq \boldsymbol{Q} \underset{\boldsymbol{M}}{\times} \boldsymbol{Q} : q \mapsto (q, q)$ the *Liouville field* on $\boldsymbol{Q}$.

Let us consider the bundle $\boldsymbol{Q}^\uparrow : U_1\boldsymbol{M} \underset{\boldsymbol{M}}{\times} \boldsymbol{Q} \to U_1\boldsymbol{M}$. A *universal connection* is a connection Ξ on $\boldsymbol{Q}^\uparrow$ such that $X \lrcorner \Xi = 0$ for every vertical vector field $X : \boldsymbol{M} \to VU_1\boldsymbol{M}$. The universal connection can also be interpreted as a family of connections on $\boldsymbol{Q} \to \boldsymbol{M}$ parametrised by observers, i.e. sections of $U_1\boldsymbol{M} \to \boldsymbol{M}$.

Definition 2.2 A connection ч on the bundle $\boldsymbol{Q}^{\uparrow}$ which is Hermitian, universal and whose curvature $R[ч]$ fulfill $R[ч] = i\Omega \otimes и$, is said to be a *quantum connection.* □

We remark that the identity $d\Omega = 0$ turns out to be equivalent to the Bianchi identity for a quantum connection ч.

Definition 2.3 A pair $(\boldsymbol{Q}, ч)$ is said to be a *quantum structure.* Two quantum structures $(\boldsymbol{Q}_1, ч_1)$, $(\boldsymbol{Q}_2, ч_2)$, are said to be *equivalent* if there exists an equivalence $f : \boldsymbol{Q}_1 \to \boldsymbol{Q}_2$ which maps $ч_1$ into $ч_2$. □

As we will see, in general not any quantum bundle admits a quantum structure. We say a quantum bundle $\boldsymbol{Q}$ to be *admissible* if there exists a quantum structure $(\boldsymbol{Q}, ч)$. We denote by $\mathcal{QB} \subset \mathcal{L}(\boldsymbol{M})$ the set of equivalence classes of admissible quantum bundles. Let $[\boldsymbol{Q}] \in \mathcal{QB}$. Then we define $\mathcal{QS}[\boldsymbol{Q}]$ to be the set of equivalence classes of quantum structures having quantum bundles in $[\boldsymbol{Q}]$. If $[\boldsymbol{Q}'] \in \mathcal{QB}$ and $[\boldsymbol{Q}] \neq [\boldsymbol{Q}']$, then $\mathcal{QS}[\boldsymbol{Q}]$ and $\mathcal{QS}[\boldsymbol{Q}']$ are clearly disjoint. So, we define

$$\mathcal{QS} := \bigsqcup_{[\boldsymbol{Q}] \in \mathcal{QB}} \mathcal{QS}[\boldsymbol{Q}] \tag{2}$$

to be the set of equivalence classes of quantum structures.

The task of the rest of the paper is to analyse the structures of $\mathcal{QB}$ and $\mathcal{QS}$. With this objective in mind, we devote the final part of this subsection to some technical results.

Lemma 2.1 *Let $\boldsymbol{U} \subset \boldsymbol{M}$ be a star–shaped trivialising neighbourhood of $\boldsymbol{Q}$. Then, $ч = ч^{\|}_{\boldsymbol{U}} + i\tau_{\boldsymbol{U}} \otimes и$ where $ч^{\|}_{\boldsymbol{U}}$ is the flat connection on the trivialisation induced in $\boldsymbol{Q}^{\uparrow}$, and $\tau_{\boldsymbol{U}}$ is a potential of Ω on a trivialisation induced in $U_1\boldsymbol{M}$. More precisely, $\tau_{\boldsymbol{U}}$ takes the form $\tau_{\boldsymbol{U}} = \tau^{\natural} + A$, where $A : \boldsymbol{M} \to T^*\boldsymbol{M}$ is a (local) form such that $2dA = F$.*

The proof of the above lemma is obtained by means of the coordinate expression $ч = d^{\lambda} \otimes \partial_{\lambda} + d^i_0 \otimes \partial^0_i + i ч_{\lambda} d^{\lambda} \otimes и$ and the expression of $R[ч]$.

Remark 2.1 The above theorem shows that there is a bijection between quantum connections and Hermitian connections on $\boldsymbol{Q} \to \boldsymbol{M}$ such that their curvature is $iF \otimes и$. Anyway, in several points of this quantum theory we need geometric objects on $\boldsymbol{Q}^{\uparrow}$ [12, 13]. □

Now, we study the change of the coordinate expression of a quantum connection ч with respect to a change of chart. Let $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ be two coordinate star–shaped open subsets of $\boldsymbol{M}$ such that $\boldsymbol{U}_1 \cap \boldsymbol{U}_2 \neq \emptyset$, and b_1, b_2 be the local bases for quantum sections induced by the choice of two corresponding trivialisations of $\boldsymbol{Q}$. Suppose that the change of base is expressed by c_{12}, or, equivalently, by f_{12}, as $b_1 = c_{12} b_2 = \exp(2\pi i f_{12}) b_2$, where $c_{12} : \boldsymbol{U}_1 \cap \boldsymbol{U}_2 \to U(1)$ and $f_{12} : \boldsymbol{U}_1 \cap \boldsymbol{U}_2 \to I\!R$. According to the above lemma, in $\boldsymbol{U}_1 \cap \boldsymbol{U}_2$ let $ч = ч^{\|}_1 + i\tau_1 \otimes и = ч^{\|}_2 + i\tau_2 \otimes и$. Then, it follows from a coordinate computation that

$$ч^{\|}_1 = ч^{\|}_2 - 2\pi i df_{12} \otimes и \qquad \tau_1 = \tau_2 + 2\pi i df_{12} \otimes и \tag{3}$$

Remark 2.2 We have introduced the quantum structures on an Einstein's general relativistic background, by analogy with the Galilei general relativistic quantum structures. In [17] an algebra of quantisable functions was introduced, and in [18] a corresponding algebra of quantum operators has been studied. This leads to a covariant prequantisation of the mechanics of a scalar particle on Einstein general relativistic spacetime (see [15] for the covariant Klein–Gordon equation). Quantisation will be the subject of future works. □

Now, we give a necessary and sufficient condition for the existence of a quantum bundle and a quantum connection. This is carried on by analogy with the Galilei's general relativistic case [32, 33, 34] and the standard geometric quantisation [8, 19]. In particular, the necessary and sufficient condition is Einstein general relativistic analogue of the Kostant–Soriau theorem in the standard geometric quantisation.

We follow a presentation of the Kostant–Souriau theorem [19, 29] given in [8]. See also [25, 33]. We recall the (not necessarily injective) group morphism $i : H^2(\boldsymbol{M}, \mathbb{Z}) \to H^2(\boldsymbol{M}, I\!R)$ induced by the inclusion $\mathbb{Z} \hookrightarrow I\!R$. Moreover, we remark that the cohomology class of the closed total form Ω depends only on the cohomology class of the electromagnetic form F.

Theorem 2.1 *The following conditions are equivalent.*

1. *There exists a quantum structure* $(\boldsymbol{Q}, \text{ч})$.

2. *The cohomology class* $[qs] \in H^2(\boldsymbol{M}, I\!R)$ *determined by the (de Rham class of the) closed form* Ω *(hence by* F*) lies in the subgroup*

$$[qs] \in i(H^2(\boldsymbol{M}, \mathbb{Z})) \subset H^2(\boldsymbol{M}, I\!R)\,.$$

Proof. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be a *good cover* of $\boldsymbol{M}$, i.e. an open cover in which any finite intersection is either empty or diffeomorphic to $I\!R^n$. Suppose that the second condition of the statement holds. Then, we observe that the morphism $i : H^2(\boldsymbol{M}, \mathbb{Z}) \to H^2(\boldsymbol{M}, I\!R)$ is given as $i([qs]) = [i(qs)]$, where $(i(qs))_{ijk} := i((qs)_{ijk})$ for each $i, j, k \in I$ with $U_i \cap U_j \cap U_k \neq \emptyset$.

So, for each $i \in I$ we can choose potentials A_i of F defined on U_i, and for each $i, j \in I$ with $U_i \cap U_j \neq \emptyset$ we can choose potentials f_{ij} of $A_i - A_j$ on $U_i \cap U_j$ such that for any $i, j, k \in I$ with $U_i \cap U_j \cap U_k \neq \emptyset$ we have $(f_{ij} + f_{jk} - f_{ik}) \in \mathbb{Z}$. Let us set $c_{ij} := \exp 2\pi i f_{ij}$. We have $c_{ij} c_{jk} = c_{ik}$, so we obtain a 1–cocycle on $\boldsymbol{M}$ which gives rise to an isomorphism class $[\boldsymbol{Q}] \in \mathcal{L}(\boldsymbol{M})$

Moreover, we have $A_i - A_j = 1/(2\pi i)\, dc_{ij}/c_{ij}$; so the forms $i(\tau^{\natural} + A_i) \otimes \text{и}$ give a global quantum connection.

Conversely, if the first condition of the statement holds, we use theorem 2.1 and equation (3) with respect to a trivialisation over the good cover. The functions f_{ij} give rise to the constant functions $f_{ij} + f_{jk} - f_{ik}$ with values in $\mathbb{Z}$, hence to a class $[qs] \in i(H^2(\boldsymbol{M}, \mathbb{Z}))$. QED

Now, we classify inequivalent quantum structure on a given classical background (which, of course, fulfills the existence condition). Let m be a mass and q a particle. We start by assuming that the existence condition is satisfied.

Assumption Q.1 We assume that the electromagnetic form F fulfills the following *integrality condition*: $[F] \in i(H^2(\boldsymbol{M}, \mathbb{Z})) \subset H^2(\boldsymbol{M}, I\!R)$.

□

The first (rather obvious) classification result shows the structure of $\mathcal{QB}$.

Theorem 2.2 *The set $\mathcal{QB} \subset \mathcal{L}(\boldsymbol{M})$ of quantum bundles compatible with Ω is the set $i^{-1}([qs]) \subset H^2(\boldsymbol{M}, \mathbb{Z})$; this set is in bijection with $\ker i \subset H^2(\boldsymbol{M}, \mathbb{Z})$.*

Let $[\boldsymbol{Q}, \text{ч}], [\boldsymbol{Q}', \text{ч}'] \in \mathcal{QS}[\boldsymbol{Q}]$, $f : \boldsymbol{Q} \to \boldsymbol{Q}'$ be a bundle equivalence and f_* be the induced map on connections. Then we have $\text{ч}' - f_*\text{ч} = -2\pi i D \otimes \text{и}$, where D is a closed form on $\boldsymbol{M}$. Moreover, $[\boldsymbol{Q}, \text{ч}] = [\boldsymbol{Q}', \text{ч}']$ if and only if $D = 1/(2\pi i)\, dc/c$, where $c : \boldsymbol{M} \to U(1)$.

Lemma 2.2 *There exists an abelian group isomorphism*

$$H^1(\boldsymbol{M}, \mathbb{Z}) \to \left\{ \frac{1}{2\pi i} \frac{dc}{c} | c : \boldsymbol{M} \to U(1) \right\} .$$

Proof. Using a procedure similar to the proof of the existence theorem we can prove that the right–hand set is isomorphic to $i(H^1(\boldsymbol{M}, \mathbb{Z}))$. A standard argument [36] shows that $i : H^1(\boldsymbol{M}, \mathbb{Z}) \to H^1(\boldsymbol{M}, I\!R)$ is an injective morphism. QED

Now, we are able to classify the (inequivalent) quantum structures having equivalent quantum bundles. In fact, the above lemma suggests that inequivalent quantum structures are parametrised by elements $[D] \in H^1(\boldsymbol{M}, I\!R)\big/H^1(\boldsymbol{M}, \mathbb{Z})$.

Theorem 2.3 *Let $[\boldsymbol{Q}] \in \mathcal{QB}$. Then the set $\mathcal{QS}[\boldsymbol{Q}]$ is in bijection with the quotient group $H^1(\boldsymbol{M}, I\!R)\big/H^1(\boldsymbol{M}, \mathbb{Z})$.*

The structure of the set $\mathcal{QS}$ is easily recovered from its definition and the above two theorems. Let us set $p : \mathcal{QS} \to \mathcal{QB} : [\boldsymbol{Q}, \text{ч}] \mapsto [\boldsymbol{Q}]$; p is a surjective map.

Theorem 2.4 *There exists a bijection $B : \mathcal{QS} \to H^1(\boldsymbol{M}, I\!R)\big/H^1(\boldsymbol{M}, \mathbb{Z}) \times \ker i$.*

Sometimes it is preferable to express the above product group in a more compact way. A standard cohomological argument [8, 36] yields the exact sequence

$$0 \to H^1(\boldsymbol{M}, I\!R)\big/H^1(\boldsymbol{M}, \mathbb{Z}) \to H^1(\boldsymbol{M}, U(1)) \stackrel{\delta_1}{\to} \ker i \to 0\,,$$

where δ_1 is the Bockstein morphism. So, for every equivalence class $[\boldsymbol{Q}] \in \ker i$ the set ${\delta_1}^{-1}([\boldsymbol{Q}])$ is in bijection with $H^1(\boldsymbol{M}, I\!R)\big/H^1(\boldsymbol{M}, \mathbb{Z})$.

Corollary 2.1 *The set of quantum structures is in bijection with the abelian group $H^1(\boldsymbol{M}, U(1))$. If $\boldsymbol{M}$ is simply connected, then there exists only one equivalence class of quantum structures.*

PROOF. The first assertion is due to the structure of the map δ_1. The last assertion follows from the natural isomorphism $H^1(\boldsymbol{M}, U(1)) \simeq \mathrm{Hom}(\pi_1(\boldsymbol{M}), U(1))$. QED

3 Examples of quantum structures

From a physical viewpoint, it is very interesting to study concrete exact solutions. The following examples are a starting point for an analysis of the classification of quantum structures on exact solutions of Einstein's general relativity.

Example 3.1 *Minkowski spacetime* is topologically trivial, hence the equivalence class of F in $H^2(\boldsymbol{M}, \mathbb{R})$ is the zero class. Therefore, the integrality condition is fulfilled, and the Minkowski spacetime admits quantum structures. Corollary 2.1 yields that there exists a unique equivalence class of quantum structures. A distinguished set of representatives of this equivalence class is provided by the trivial quantum bundle together with quantum connections built by means of the natural flat connection, $\tau^\natural$, and one global potential of F. □

Example 3.2 *Schwartzschild spacetime* has the topology of $\mathbb{R} \times (\mathbb{R}^3 \backslash \{0\})$ [10, 23], hence it is simply connected. Being $F = 0$, the integrality condition is fulfilled, and Corollary 2.1 yields the existence of a unique equivalence class of quantum structures. A distinguished representative of this equivalence class is provided by the trivial quantum bundle together with the quantum connection built by means of the natural flat connection and $\tau^\natural$.

We observe that the same considerations hold for *Kruskal spacetime*, yielding the same results. □

Example 3.3 *Dirac's monopole* is a particular kind of electromagnetic field F built starting from Minkowski spacetime $\boldsymbol{M}$.

We define an *inertial motion* $s \subset \boldsymbol{M}$ to be a positively oriented time–like one–dimensional affine subspace of $\boldsymbol{M}$. Let us denote by $\bar{\boldsymbol{M}}$ the vector space associated with $\boldsymbol{M}$. Then, by the natural isomorphism $T\boldsymbol{M} \simeq \boldsymbol{M} \times \bar{\boldsymbol{M}}$, we have the natural isomorphism $U_1\boldsymbol{M} \simeq \boldsymbol{M} \times \mathbb{T}^* \otimes \bar{\boldsymbol{M}}_t$, where $\bar{\boldsymbol{M}}_t$ denotes the subset of time–like vectors of $\bar{\boldsymbol{M}}$. We say an *inertial observer* o to be an observer which yields a constant map $\boldsymbol{M} \to \mathbb{T}^* \otimes \bar{\boldsymbol{M}}_t$ through the above identification. Of course, the flow of an inertial observer consists of inertial motions.

–We assume an inertial motion $s \subset \boldsymbol{M}$.

Such a motion induces an inertial observer o by means of the translations of $\boldsymbol{M}$, hence a splitting $\boldsymbol{M} \to s \times \boldsymbol{P}$, where $\boldsymbol{P}$ is the set of (inertial) motions $s' \subset \boldsymbol{M}$ fulfilling $j_1 s' = o \circ s'$. The above splitting carries the scaled metric of $\boldsymbol{M}$ into a scaled metric on $s \times \boldsymbol{P}$. It turns out that $\boldsymbol{P}$ is endowed with the structure of an Euclidean vector space, the 0 representing the motion s. Let us set $\boldsymbol{P}' := \boldsymbol{P} \setminus \{0\}$; we have the isometric splitting $\boldsymbol{P}' \to \mathbb{L} \times S^2$, where $\mathbb{L}$ represents the distance from the origin and S^2 is the space of directions which is diffeomorphic to the unit sphere in $\boldsymbol{P}$ and is endowed with a natural non–scaled metric. The scaled multiples of the volume form ν on S^2 are natural candidates for the electromagnetic field.

– We assume a mass m and a charge q. Moreover, we assume the magnetic field

$$F := k\nu : S^2 \to \overset{2}{\wedge} T^* S^2 \ , \tag{4}$$

where k is a real constant. The coordinate expression of F with respect to polar coordinates turns out to be $F = k \sin\vartheta\, d^\vartheta \wedge d^\varphi$. Of course, we have $dF = 0$.

We have

$$[\Omega] = [\frac{1}{2}F] = [\frac{1}{2}\, k\, \nu]\, .$$

A computation [9, p.164] shows that if $k \in \mathbb{Z}$ then $[\Omega]$ fulfills the integrality condition. Being spacetime topologically equivalent to $I\!R \times (I\!R^3 \setminus \{0\})$, corollary 2.1 yields that for any $k \in \mathbb{Z}$ there exists a unique equivalence class of quantum structures compatible with F.

It is interesting to note that if $k, k' \in \mathbb{Z}$, then the respective quantum bundles are not isomorphic. In particular, if $k \neq 0$, then the class of quantum bundles compatible with F is not the trivial class. So, this is a first example of non–trivial quantum structure.

Following [9], we can give a physical interpretation to k. In particular, we can set $f = \mu\nu$, where $\mu \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$ is assumed to be the *magnetic charge* of the monopole. So, $k = (q\mu)/(\hbar c)$.

We remark that there exists a purely gravitational example of such a non–trivial situation provided by the nonrelativistic limit of the Taub–NUT solution, and leading to quantisation of mass [5]. □

Example 3.4 The *Aharonov–Bohm effect* is produced in a Minkowski spacetime $\boldsymbol{M}$ by a solenoidal magnetic field in the region where the magnetic field vanishes. In this region the vector potential can be different from zero, producing effects at the quantum level. The reader can compare our formalisation with the discussions in [11, 37].

–We assume a one–parameter family of inertial motions $\{s_t\}_{t \in I\!R}$, fulfilling the relation $s_t = s_0 + tv$.

Each s_t induces the same inertial observer o. Hence, we have the splitting $\boldsymbol{M} \to s_0 \times \boldsymbol{P}$. The family $\{s_t\}$ is represented by a line $\boldsymbol{S} \subset \boldsymbol{P}$ passing through the origin in $\boldsymbol{P}$. This line is the model of our ideal solenoid. Let us set $\boldsymbol{P}' := \boldsymbol{P} \setminus \boldsymbol{S}$; we have the isometric splitting $\boldsymbol{P}' \to \boldsymbol{S} \times \mathbb{L} \times S^1$, where $\mathbb{L}$ represents the distance from the line $\boldsymbol{S}$ and S^1 is the space of directions, which is diffeomorphic to a unit circle in $\boldsymbol{P}$ and is endowed with a natural non–scaled metric. We set $\boldsymbol{M}' := s_0 \times \boldsymbol{P}'$.

– We assume a mass m and a charge q. Moreover, we assume the magnetic field $F = 0$ on $\boldsymbol{M}'$.

Of course, the spacetime $\boldsymbol{M}'$ endowed with the induced metric from $\boldsymbol{M}$ and the electromagnetic field $F = 0$ fulfills the integrality condition. But the cohomologies of $\boldsymbol{M}'$ and S^1 are isomorphic, hence this spacetime admits $H^1(\boldsymbol{M}', I\!R)/H^1(\boldsymbol{M}', \mathbb{Z}) \simeq U(1)$ inequivalent quantum structures. A direct computation shows that $\ker i = \{0\}$. Then, we have a unique equivalence class of admissible quantum bundles, represented by the trivial bundle $\boldsymbol{M}' \times \mathbb{C} \to \boldsymbol{M}'$, and $U(1)$ inequivalent quantum connections.

We consider the (real) multiples of the volume form ν on S^1 $k\nu : S^1 \to T^*S^1$, with coordinate expression $k\nu = kd^\theta$. The quantum connection

$$ч[k] := ч^{\|} + i(\tau^\natural + k\nu) \otimes и$$

is an admissible quantum connection. Moreover, $ч[k]$ is equivalent to $ч[k']$ if and only if $k' - k$ is an integer. Hence, we have described the set of inequivalent quantum structures on $\boldsymbol{M}'$. We can interpret $k\nu$ as the potential of $F = 0$, and set $k = (q\mu)/(\hbar c)$ as in the above example, μ being the magnetic charge of the solenoid. □

Example 3.5 The *Kerr–Newman spacetime* is the unique axisymmetric static exact solution of Einstein equations in the vacuum [10, 23].

– We assume a one–parameter family of inertial motions $\{s_t\}_{t \in I\!R}$, fulfilling the relation $s_t = s_0 + tv$.

As in the previous example, we have the isomorphism $\boldsymbol{M} \to s_0 \times \boldsymbol{P}$, and, in this case, the line $\boldsymbol{S}$ stands for the position of the symmetry axis. Moreover, here we make use of a coordinate system (t, r, θ, ϕ) on $\boldsymbol{M}' := s_0 \times \boldsymbol{P}'$ provided by a diffeomorphism $\boldsymbol{P}' \simeq I\!R_+ \times]0, \pi[\times S^1$, where S^1 is the unit circle in $I\!R^2$.

– We assume a mass M, a charge Q, and an *angular momentum* $S \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$.

The Kerr–Newman spacetime is determined uniquely by the three above constants together with the gravitational coupling constant κ [23]. We set $a := S/(Mc)$, $b := \sqrt{\kappa} Q/c^2$, $p = \kappa M/c^2$, $\Delta = r^2 - 2pr + a^2 + b^2$, $\rho^2 = r^2 + a^2 \cos^2\theta$.

– We assume the metric

$$g = -\frac{\Delta}{\rho^2}(d^t - a\sin^2\theta d^\phi)^2 + \frac{\sin^2\theta}{\rho^2}\, d^r \otimes d^r + \rho^2\, d^\theta \otimes d^\theta$$

and the electromagnetic field

$$\begin{aligned} F =& b\rho^{-4}(r^2 - a^2\cos^2\theta)\, d^r \wedge (d^t - a\sin^2\theta\, d^\phi) + \\ & 2b\rho^{-4} ar\cos\theta\sin\theta\, d^\theta \wedge ((r^2 + a^2)\, d^\phi - a\, d^t) \end{aligned}$$

(of course, F and g are globally defined).

The electromagnetic field F has a global potential

$$A = -\frac{br}{\rho^2}\,(d^t - a\sin^2\theta\, d^\phi)$$

hence the integrality condition is fulfilled. The topological analysis of $\boldsymbol{M}$ leads to the same conclusions as the above example. Thus, we have the trivial quantum bundle as the representative of the class of admissible quantum bundles. Moreover,

$$\text{ч}[k] := \text{ч}^{\|} + i(\tau^{\natural} + A + k\nu)\otimes\text{и}$$

is an admissible quantum connection. Of course, $\text{ч}[k]$ is equivalent to $\text{ч}[k']$ if and only if $k' - k$ is an integer. Hence, we have described the set of inequivalent quantum structures on $\boldsymbol{M}'$.

We do not know any significant physical interpretation of the potential $k\nu$. If $k \neq 0$, then such a term would imply quantum effects of the Aharonov–Bohm type nearby a charged rotating black hole. □

Remark 3.1 We have no experimental evidence for the Dirac monopole example. Conversely, the Aharonov–Bohm effect was verified with electrons, while our theory deals only with scalar particles. The computations of the last example seem to indicate the existence of quantum effects of the Aharonov–Bohm type nearby a charged rotating black hole. We do not know if there is any possibility of experimental verification of such an effect. □

References

[1] C. Albert: *Le théorème de reduction de Marsden–Weinstein en géométrie cosymplectique et de contact*, J. Geom. Phys., p.627 ff., V.6, n.4, 1989.

[2] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer: Deformation theory and quantization I and II, *Ann. of Phys.* **3** (1978), 61–151.

[3] D. Canarutto, A. Jadczyk, M. Modugno: Quantum mechanics of a spin particle in a curved spacetime with absolute time, *Rep. Math. Phys.*, **36**, 1 (1995), 95–140.

[4] C. Duval, G. Burdet, H. P. Künzle, M. Perrin: Bargmann structures and Newton–Cartan theory, *Phys. Rev. D,* **31**, n. 8 (1985), 1841–1853.

[5] C. Duval, G. Gibbons, P. Horváthy: Celestial mechanics, conformal structures, and gravitational waves, *Phys. Rev. D,* **43**, 3907 (1991).

[6] M. de Leon, M. Saralegi: *Cosymplectic reduction for singular momentum maps*, J. Phys. A **26** (1993) 5033–5043.

[7] C. Duval, H. P. Künzle: Minimal gravitational coupling in the Newtonian theory and the covariant Schrödinger equation, *Gen. Rel. Grav.,* **16**, n. 4 (1984), 333–347.

[8] P. L. Garcia: Cuantificion geometrica, *Memorias de la R. Acad. de Ciencias de Madrid,* **XI**, Madrid, 1979.

[9] M. Göckeler, T. Schücker: *Differential geometry, gauge theories and gravity,* Cambridge Univ. Press, Cambridge, 1987.

[10] S. Hawking, G. Ellis: *The large scale structure of space–time,* Cambridge University Press, Cambridge, 1973.

[11] P. Horvathy, J. Kollar: The nonabelian Bohm-Aharonov effect in geometric quantisation *Class. and Quant. Grav.* **1**, n. 6 (1984), 61–66.

[12] A. Jadczyk, M. Modugno: A scheme for Galilei general relativistic quantum mechanics, in *Proceedings of the* 10^{th} *Italian Conference on General Relativity and Gravitational Physics,* World Scientific, New York, 1993.

[13] M. Modugno: *Galilei General Relativistic Quantum Mechanics,* 1999, book in preparation.

[14] J. Janyška: Remarks on symplectic and contact 2–forms in relativistic theories, *Boll. Un. Mat. Ital.,* **7** (1994), 587-616.

[15] J. Janyška: *Natural Lagrangians for quantum structures over 4–dimensional spaces*, Rend. di Mat., S VII, Vol 18, Roma (1998), 623–648. *

[16] J. Janyška, M. Modugno: Classical particle phase space in general relativity, in *Proc. "Diff. Geom. and Appl."*, Brno, 1995.

[17] J. Janyška, M. Modugno: Quantisable functions in general relativity, in *Proc. "Diff. Oper. and Math. Phys."*, Coimbra; World Scientific, 1995.

[18] J. Janyška, M. Modugno: On quantum vector fields in general relativistic quantum mechanics, in *Proc. Conf. Diff. Geom.*, Sibiu, Romania, Sept. 1997.

[19] B. Kostant: Quantization and unitary representations, in *Lectures in Modern Analysis and Applications III*, Springer–Verlag, **170** (1970), 87–207.

[20] K. Kuchař: Gravitation, geometry and nonrelativistic quantum theory, *Phys. Rev. D,* **22**, n. 6 (1980), 1285–1299.

[21] H. P. Künzle: General covariance and minimal gravitational coupling in Newtonian spacetime, in A. Prastaro (ed.) *Geometrodynamics Proceedings* Tecnoprint, Bologna 1984, 37–48.

[22] P. Libermann, Ch. M. Marle: *Symplectic Geometry and Analytical Mechanics,* Reidel Publ., Dordrecht, 1987.

[23] W. Misner, K. Thorne, J. Wheeler: *Gravitation,* Freeman, S. Francisco, 1978.

[24] M. Modugno, A. M. Vinogradov: Some variations on the notions of connection, *Annali di Matematica pura ed Applicata* (IV), Vol. CLXVII (1994), 33–71.

[25] M. Modugno, R. Vitolo: Quantum connection and Poincaré–Cartan form, in: G. Ferrarese (ed.) *Proc. Conf. in honour of A. Lichnerowicz*, Frascati, october 1995; Pitagora, Bologna.

[26] E. Prugovečki: *Principles of quantum general relativity*, World Scientific, Singapore, 1995.

[27] E. Schmutzer, J. Plebanski: Quantum mechanics in non–inertial frames of reference, *Fortschritte der Physik* **25** (1977), 37–82.

[28] J. Sniaticki: *Geometric quantization and quantum mechanics*, Springer–Verlag, New York 1980.

[29] J.–M. Souriau: *Structures des systèmes dynamiques*, Dunod, Paris 1970.

[30] A. Trautman: Comparison of Newtonian and relativistic theories of space–time, in *Perspectives in geometry and relativity (Essays in Honour of V. Hlavaty),* n. 42, Indiana Univ. press, 1966, 413–425.

[31] W. M. Tulczyjew: An intrinsic formulation of nonrelativistic analytical mechanics and wawe mechanics, *J. Geom. Phys.,* **2**, n. 3 (1985), 93–105.

[32] R. Vitolo: Bicomplessi lagrangiani ed applicazioni alla meccanica relativistica classica e quantistica, *PhD Thesis,* Florence, 1996.

[33] R. Vitolo: Quantum structures in general relativistic theories, *Proc. of the Ital. Conf on Gen. Rel. and Grav. Phys.,* Roma, 1996; World Scientific, Singapore, 1998.

[34] R. Vitolo: Quantum structures in Galilei general relativity *Ann. de l'Inst. H. Poincaré,* **70**, n. 3 (1999).

[35] R. Vitolo: Quantising a rigid body, to appear in *Proc. 'Diff. Geom. and Appl.'*, Brno, 1998.

[36] R. O. Wells: *Differential Analysis on Complex Manifolds,* GTM, n. 65, Springer–Verlag, Berlin, 1980.

[37] N. Woodhouse: *Geometric quantization,* Second Ed., Clarendon Press, Oxford 1992.