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On different geometric formulations of Lagrangian formalism

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Abstract

We consider two geometric formulations of Lagrangian formalism on fibred manifolds: Krupka's theory of finite order variational sequences, and Vinogradov's infinite order variational sequence associated with the $\mathcal{C}$–spectral sequence. On one hand, we show that the direct limit of Krupka's variational bicomplex is a new infinite order variational bicomplex which yields a new infinite order variational sequence. On the other hand, by means of Vinogradov's $\mathcal{C}$–spectral sequence, we provide a new finite order variational sequence whose direct limit turns out to be the Vinogradov's infinite order variational sequence. Finally, we provide an equivalence of the two finite order and infinite order variational sequences up to the space of Euler–Lagrange morphisms.

Key words: Fibred manifold, jet space, infinite order jet space, variational bicomplex, variational sequence, spectral sequence.
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Introduction

The theory of variational bicomplexes can be regarded as the natural geometrical setting for the calculus of variations [1, 2, 10, 11, 15, 19, 20, 21, 22, 23, 24]. The geometric objects which appear in the calculus of variations find a place on the vertices of a variational bicomplex, and are related by the morphisms of the bicomplex. Such morphisms are closely related to the differential of forms. Moreover, the global inverse problem is solved in this context.

The purpose of this paper is to compare Krupka's finite order formulation [10] to the infinite order formulation by Vinogradov [23, 24].

Krupka's finite order variational sequence is produced when one quotients the de Rham sequence on a finite order jet space by means of an intrinsically defined subsequence. So, the morphisms of this bicomplex are either the differential of forms, or inclusions, or quotient morphisms. A finite order formulation of variational bicomplexes can help in keeping trace of the order of the geometric objects involved at each vertex of the bicomplex. But this yields several technical difficulties. For an intrinsic analysis of this theory, based on the structure form on jets [13] and the first variation formula [9], see [28].

The formulation of Vinogradov is carried on by means of the $\mathcal{C}$–spectral sequence. This is a very general framework, by which one can formulate the variational sequences also in the case of the spaces of infinite jets of m dimensional submanifolds of a given $m+n$ dimensional manifold. Moreover, $\mathcal{C}$–spectral sequences play an important role in the theory of ordinary and partial differential equations, and in quantum mechanics and field theory. Here, we consider the variational sequence associated with the $\mathcal{C}$–spectral sequence of the infinite order de Rham exact sequence. Roughly speaking, the infinite order de Rham exact sequence is made by forms on jet spaces of any order. This is a wiewpoint that allows to skip several hard technical difficulties. In fact, one has not to worry about the order of the objects or the operators. The relationship between Tulczyjew's and Vinogradov's formulations has been analysed in [4].

Here, we give a new finite order formulation of variational sequences using the $\mathcal{C}$–spectral sequence on finite order jet spaces. The direct limit of this finite order $\mathcal{C}$–spectral sequence turns out to be Vinogradov's infinite order $\mathcal{C}$–spectral sequence.

Then, we evaluate the direct limit of Krupka's variational bicomplex, finding a new infinite order variational sequence.

Finally, we do a comparison of both finite and infinite order variational sequences finding that they are isomorphic up to the space of Euler–Lagrange morphisms.

So, the logical scheme of this paper is summarised by the following diagram

$$\begin{tikzcd}
\begin{array}{c}\text{New infinite–order}\\ \text{formulation}\end{array} \arrow[r, "\approx"] & \begin{array}{c}\text{Vinogradov's infinite–order}\\ \text{formulation}\end{array} \\
\begin{array}{c}\text{Krupka's finite–order}\\ \text{formulation}\end{array} \arrow[u, "\underrightarrow{\lim}"] \arrow[r, "\approx"] & \begin{array}{c}\text{New finite–order}\\ \text{formulation}\end{array} \arrow[u, "\underrightarrow{\lim}"']
\end{tikzcd}$$

A final discussion has to be devoted to the language used in the paper. While Krupka's approach is carried on by means of the language of sheaves (see, for example, [29]), Vinogradov's approach uses an algebraic language (see [8] and references therein). Here, in order to compare the above approaches with a unique language, we found easier to left Krupka's approach unchanged and to use a presheaf approach for the $\mathcal{C}$–spectral sequence. In fact, the $\mathcal{C}$–spectral sequence is defined in the category of differential groups (Appendix B), but, in our case, it can be carried on easily to presheaves. Of course, the algebraic language is a very powerful and natural tool, and in the next future much more could be said about finite order $\mathcal{C}$–spectral sequences, for example in the case of jets of submanifolds.

We observe that a basic introduction to spectral sequences is provided in Appendix B, in order to make the paper self–contained.

We end the introduction with some mathematical conventions. In this paper, manifolds are connected and C^∞, and maps between manifolds are C^∞. Morphisms of fibred manifolds (and hence bundles) are morphisms over the identity of the base manifold, unless otherwise specified.

We make use of definitions and results on presheaves and sheaves from [29]. In particular, we are concerned only with (pre)sheaves of $I\!R$–vector spaces, hence '(pre)sheaf morphism' stands for morphism of (pre)sheaves of $I\!R$–vector spaces. We denote by $\mathcal{S}_U$ the set of sections of a (pre)sheaf $\mathcal{S}$ over a topological space X defined on the open subset $U \subset X$. We recall that a sequence of (pre)sheaves over X is said to be exact if it is locally exact (see [29] for a more precise definition). If $\mathcal{A}$, $\mathcal{B}$ are two sub(pre)sheaves of a sheaf $\mathcal{S}$, then the wedge product $\mathcal{A}\wedge\mathcal{B}$ is defined to be the sub(pre)sheaf of sections of $\overset{2}{\wedge}\mathcal{S}$ generated by wedge products of sections of $\mathcal{A}$ and $\mathcal{B}$.

We recall that a sheaf $\mathcal{S}$ over X is said to be *soft* if each section defined on a closed subset $C \subset X$ can be extended to a section defined on any open subset U such that $C \subset U$. Moreover, $\mathcal{S}$ is said to be *fine* if it admits a partition of unity. A fine sheaf is also a soft sheaf. The sheaf of sections of a vector bundle is a fine sheaf, hence a soft sheaf.

Let $\{\mathcal{S}_n\}_{n\in\mathbb{N}}$ be a family of (pre)sheaves and $\{\iota^m_n : \mathcal{S}_n \to \mathcal{S}_m\}_{n,m\in\mathbb{N},n\leq m}$ be a family of injective (pre)sheaf morphisms such that, for all $n, m, p \in \mathbb{N}$, $n \leq m \leq p$, we have $\iota^p_m \circ \iota^m_n = \iota^p_n$ and $\iota^n_n = \mathrm{id}_{\mathcal{S}_n}$. We say $\{\mathcal{S}_n\}$ to be an *injective system*. We define the *direct limit* of the injective system to be the (pre)sheaf

$$\mathcal{S} := \bigsqcup_{n\in\mathbb{N}} \mathcal{S}_n / \sim ,$$

where $\sim$ is the equivalence relation defined as follows. For each $s \in \mathcal{S}_n$ and $s' \in \mathcal{S}_{n'}$, if $n \leq n'$, then $s \sim s'$ if and only if $\iota^{n'}_n(s) = s'$.

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1 Jet spaces

In this section we recall some facts on jet spaces. We start with the definition of jet space, then we introduce the contact maps. We study the natural sheaves of forms on jet spaces which arise from the fibring and the contact maps. Finally, we introduce the horizontal and vertical differential of forms on jet spaces.

Jet spaces

Our framework is a fibred manifold

$$\pi : \boldsymbol{Y} \to \boldsymbol{X} ,$$

with $\dim \boldsymbol{X} = n$ and $\dim \boldsymbol{Y} = n + m$.

We deal with the tangent bundle $T\boldsymbol{Y} \to \boldsymbol{Y}$, the tangent prolongation $T\pi : T\boldsymbol{Y} \to T\boldsymbol{X}$ and the vertical bundle $V\boldsymbol{Y} := \ker T\pi \to \boldsymbol{Y}$.

Moreover, for $0 \le r$, we are concerned with the r–th jet space $J_r\boldsymbol{Y}$; in particular, we set $J_0\boldsymbol{Y} \equiv \boldsymbol{Y}$. We recall the natural fibrings

$$\pi^r_s : J_r\boldsymbol{Y} \to J_s\boldsymbol{Y} , \qquad \pi^r : J_r\boldsymbol{Y} \to \boldsymbol{X} ,$$

and the affine bundle

$$\pi^r_{r-1} : J_r\boldsymbol{Y} \to J_{r-1}\boldsymbol{Y}$$

associated with the vector bundle

$$\odot^r T^*\boldsymbol{X} \underset{J_{r-1}\boldsymbol{Y}}{\otimes} V\boldsymbol{Y} \to J_{r-1}\boldsymbol{Y} ,$$

for $0 \le s \le r$. A detailed account of the theory of jets can be found in [13, 11, 17].

Charts on $\boldsymbol{Y}$ adapted to the fibring are denoted by (x^λ, y^i). Greek indices $\lambda, \mu, \dots$ run from 1 to n and label base coordinates, Latin indices $i, j, \dots$ run from 1 to m and label fibre coordinates, unles otherwise specified. We denote by $(\partial_\lambda, \partial_i)$ and (d^λ, d^i), respectively, the local bases of vector fields and 1–forms on $\boldsymbol{Y}$ induced by an adapted chart.

We denote multi–indices of dimension n by underlined latin letters such as $\underline{p} = (p_1, \dots, p_n)$, with $0 \le p_1, \dots, p_n$; by identifying the index λ with a multi–index according to

$$\lambda \simeq (p_1, \dots, p_i, \dots, p_n) \equiv (0, \dots, 1, \dots, 0) ,$$

we can write

$$\underline{p} + \lambda = (p_1, \dots, p_i + 1, \dots, p_n) .$$

We also set $|\underline{p}| := p_1 + \dots + p_n$ and $\underline{p}! := p_1! \dots p_n!$.

The charts induced on $J_r\boldsymbol{Y}$ are denoted by $(x^0, y^i_{\underline{p}})$, with $0 \le |\underline{p}| \le r$; in particular, if $|\underline{p}| = 0$, then we set $y^i_{\underline{0}} \equiv y^i$. The local vector fields and forms of $J_r\boldsymbol{Y}$ induced by the fibre coordinates are denoted by $(\partial^{\underline{p}}_i)$ and $(d^i_{\underline{p}})$, $0 \le |\underline{p}| \le r, 1 \le i \le m$, respectively.

Contact maps

A fundamental role is played in this paper by the "contact maps" on jet spaces (see [13]). Namely, for $1 \leq r$, we consider the natural injective fibred morphism over $J_r\boldsymbol{Y} \to J_{r-1}\boldsymbol{Y}$

$$\text{д}_r : J_r\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X} \to TJ_{r-1}\boldsymbol{Y} \,,$$

and the complementary surjective fibred morphism

$$\vartheta_r : J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} TJ_{r-1}\boldsymbol{Y} \to VJ_{r-1}\boldsymbol{Y} \,,$$

whose coordinate expression are

$$\begin{aligned} \text{д}_r &= d^\lambda \otimes \text{д}_{r\lambda} = d^\lambda \otimes (\partial_\lambda + y^j_{\underline{p}+\lambda} \partial^{\underline{p}}_j) \,, \qquad 0 \leq |\underline{p}| \leq r-1, \\ \vartheta_r &= \vartheta^j_{\underline{p}} \otimes \partial^{\underline{p}}_j = (d^j_{\underline{p}} - y^j_{\underline{p}+\lambda} d^\lambda) \otimes \partial^{\underline{p}}_j \,, \qquad 0 \leq |\underline{p}| \leq r-1 \,. \end{aligned}$$

We stress that

(1) $$\text{д}_r \lrcorner \vartheta_r = \vartheta_r \lrcorner \text{д}_r = 0$$

(2) $$(\vartheta_r)^2 = \vartheta_r \qquad\qquad (\text{д}_r)^2 = \text{д}_r$$

The transpose of the map ϑ_r is the injective fibred morphism over $J_r\boldsymbol{Y} \to J_{r-1}\boldsymbol{Y}$

$$\vartheta^*_r : J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} V^*J_{r-1}\boldsymbol{Y} \to T^*J_{r-1}\boldsymbol{Y} \,.$$

We have the remarkable vector subbundle

(3) $$\operatorname{im} \vartheta^*_r \subset J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*J_{r-1}\boldsymbol{Y} \subset T^*J_r\boldsymbol{Y} \;,$$

and, for $0 \leq t \leq s \leq r$, the fibred inclusions

(4) $$J_r\boldsymbol{Y} \underset{J_t\boldsymbol{Y}}{\times} \operatorname{im} \vartheta^*_t \subset J_r\boldsymbol{Y} \underset{J_s\boldsymbol{Y}}{\times} \operatorname{im} \vartheta^*_s \subset \operatorname{im} \vartheta^*_r \;.$$

The above vector subbundle $\operatorname{im} \vartheta^*_r$ yields the splitting [13]

(5) $$J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*J_{r-1}\boldsymbol{Y} = \left(J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*\boldsymbol{X} \right) \oplus \operatorname{im} \vartheta^*_r \;.$$

Distinguished sheaves of forms

We are concerned with some distinguished sheaves of forms on jet spaces.

Remark 1.1. The manifold $\boldsymbol{Y}$ is a differentiable retract of $J_r\boldsymbol{Y}$, hence the de Rham cohomologies of $\boldsymbol{Y}$ and $J_r\boldsymbol{Y}$ are isomorphic. Therefore, we reduce (pre)sheaves on $J_r\boldsymbol{Y}$ to sheaves on $\boldsymbol{Y}$ by considering for each (pre)sheaf $\mathcal{S}$ on $J_r\boldsymbol{Y}$ the (pre)sheaf induced by $\mathcal{S}$ by restricting to the tube topology on $J_r\boldsymbol{Y}$, i.e. , the topology generated by open sets of the kind $(\pi^r_0)^{-1}(\boldsymbol{U})$, with $\boldsymbol{U} \subset \boldsymbol{Y}$ open in $\boldsymbol{Y}$. So, from now on, the (pre)sheaves of forms on $J_r\boldsymbol{Y}$ and the related sub(pre)sheaves will be considered as (pre)sheaves over the topological space $\boldsymbol{Y}$ of the above kind. $\square$

Let $0 \le k$.

1. First of all, for $0 \le r$, we consider the standard sheaf $\overset{k}{\Lambda}_r$ of *k–forms* on $J_r\boldsymbol{Y}$

$$\alpha : J_r\boldsymbol{Y} \to \overset{k}{\wedge} T^* J_r\boldsymbol{Y} \,.$$

2. Then, for $0 \le s \le r$, we consider the sheaves $\overset{k}{\mathcal{H}}_{(r,s)}$ and $\overset{k}{\mathcal{H}}_r$ of *horizontal forms*, i.e. of local fibred morphisms over $J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$ and $J_r\boldsymbol{Y} \to \boldsymbol{X}$ of the type

$$\alpha : J_r\boldsymbol{Y} \to \overset{k}{\wedge} T^* J_s\boldsymbol{Y} \qquad \text{and} \qquad \beta : J_r\boldsymbol{Y} \to \overset{k}{\wedge} T^* \boldsymbol{X} \,,$$

respectively. In coordinates, if $0 < k \le n$, then

$$\alpha = \alpha^{\underline{p}_1 \dots \underline{p}_h}_{i_1 \dots i_h}{}_{\lambda_{h+1} \dots \lambda_k} \, d^{i_1}_{\underline{p}_1} \wedge \dots \wedge d^{i_h}_{\underline{p}_h} \wedge d^{\lambda_{h+1}} \wedge \dots \wedge d^{\lambda_k} \,,$$
$$\beta = \beta_{\lambda_1 \dots \lambda_k} \, d^{\lambda_1} \wedge \dots \wedge d^{\lambda_k} \,;$$

if $k > n$, then

$$\alpha = \alpha^{\underline{p}_1 \dots \underline{p}_{k-n+l}}_{i_1 \dots i_{k-n+l}}{}_{\lambda_{l+1} \dots \lambda_n} \, d^{i_1}_{\underline{p}_1} \wedge \dots \wedge d^{i_{k-n+l}}_{\underline{p}_{k-n+l}} \wedge d^{\lambda_{l+1}} \wedge \dots \wedge d^{\lambda_n} \,,$$

Here, the coordinate functions are sections of $\overset{0}{\Lambda}_r$, and the indices'range is $0 \le |\underline{p}_j| \le s$, $0 \le h \le k$ and $0 \le l \le n$. We remark that, in the coordinate expression of α, the indices λ_j are suppressed if $h = k$ or $l = n$, and the indices ${}^{\underline{p}_j}_{i_j}$ are suppressed if $h = 0$.

Clearly $\overset{k}{\mathcal{H}}_{(r,r)} = \overset{k}{\Lambda}_r$ and $\overset{k}{\mathcal{H}}_r = 0$ for $k > n$.

If $0 \le q \le r$ and $0 \le t \le s \le r$, then pull–back by π^r_q yields the sheaf inclusions

$$\overset{k}{\mathcal{H}}_q \simeq \pi^{r*}_q \overset{k}{\mathcal{H}}_q \subset \overset{k}{\mathcal{H}}_r \subset \overset{k}{\mathcal{H}}_{(r,t)} \subset \overset{k}{\mathcal{H}}_{(r,s)} \subset \overset{k}{\Lambda}_r \,,$$
$$\overset{k}{\Lambda}_q \simeq \pi^{r*}_q \overset{k}{\Lambda}_q \subset \overset{k}{\mathcal{H}}_{(r,q)} \subset \overset{k}{\Lambda}_r \,.$$

The above inclusions are proper inclusions if $t < s < r$ and $q < r$. Indeed, not all sections of the pull–back of a bundle (like $J_r\boldsymbol{Y} \underset{J_s\boldsymbol{Y}}{\times} T^* J_s\boldsymbol{Y}$) are the pull–back of some section of the bundle itself. In fact, we deal with two different operations: pull–back of bundles and pull–back of sections (forms).

3. For $0 \le s < r$, we consider the subsheaf $\overset{k}{\mathcal{C}}_{(r,s)} \subset \overset{k}{\mathcal{H}}_{(r,s)}$ of *contact* forms, i.e. of local fibred morphisms over $J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$ of the type

$$\alpha : J_r\boldsymbol{Y} \to \overset{k}{\wedge} \operatorname{im} \vartheta^*_{s+1} \subset \overset{k}{\wedge} T^* J_s\boldsymbol{Y} \,.$$

Due to the injectivity of ϑ^*_{s+1}, the subsheaf $\overset{k}{\mathcal{C}}_{(r,s)}$ turns out to be the sheaf of local fibred morphisms $\alpha \in \overset{k}{\mathcal{H}}_{(r,s)}$ which factorise as $\alpha = \overset{k}{\wedge}\vartheta^*_{s+1} \circ \tilde{\alpha}$, through the composition

$$J_r\boldsymbol{Y} \xrightarrow{\tilde{\alpha}} J_{s+1}\boldsymbol{Y} \underset{J_s\boldsymbol{Y}}{\times} \overset{k}{\wedge} V^* J_s\boldsymbol{Y} \xrightarrow{\overset{k}{\wedge}\vartheta_{s+1}} \overset{k}{\wedge} T^* J_s\boldsymbol{Y} \,.$$

Thus, $\alpha \in \overset{k}{\mathcal{C}}_{(r,s)}$ if and only if its coordinate expression is of the type

$$\alpha = \alpha^{\underline{p}_1 \dots \underline{p}_k}_{i_1 \dots i_k}\, \vartheta^{i_1}_{\underline{p}_1} \wedge \ldots \wedge \vartheta^{i_k}_{\underline{p}_k} \qquad 0 \le |\underline{p}_1|, \ldots, |\underline{p}_k| \le s \,,$$

with $\alpha^{\underline{p}_1 \dots \underline{p}_k}_{i_1 \dots i_k} \in \overset{0}{\Lambda}_r$.
If $r \le r'$, $s \le s'$ and $0 \le s < r$, $0 \le s' < r'$ then we have the inclusions (see (3) and (4))

$$\overset{k}{\mathcal{C}}_{(r,s)} \subset \overset{k}{\mathcal{C}}_{(r',s')} \,.$$

4. Furthermore, we consider the subsheaf $\overset{k}{\mathcal{H}}{}^P_r \subset \overset{k}{\mathcal{H}}_r$ of local fibred morphisms $\alpha \in \overset{k}{\mathcal{H}}_r$ such that α is a polynomial fibred morphism over $J_{r-1}\boldsymbol{Y} \to \boldsymbol{X}$ of degree k. Thus, in coordinates, $\alpha \in \overset{k}{\mathcal{H}}{}^P_r$ if and only if $\alpha_{\lambda_1,\dots,\lambda_k} : J_r\boldsymbol{Y} \to I\!R$ is a polynomial map of degree k with respect to the coordinates $y^i_{\underline{p}}$, with $|\underline{p}| = r$.

5. Finally, we consider the subsheaf $\overset{k}{\mathcal{C}}_r \subset \overset{k}{\mathcal{C}}_{(r+1,r)}$ of local fibred morphisms $\alpha \in \overset{k}{\mathcal{C}}_{(r+1,r)}$ such that $\tilde{\alpha}$ projects down on $J_r\boldsymbol{Y}$. Thus, in coordinates, $\alpha \in \overset{k}{\mathcal{C}}_r$ if and only if $\alpha^{\underline{p}_1 \dots \underline{p}_k}_{i_1 \dots i_k} \in \overset{0}{\Lambda}_r$.

Main splitting

The maps $д_r$ and ϑ_r induce two important derivations of degree 0 (see [17]), namely the interior products by $д_r$ and ϑ_r

$$i_h \equiv i_{д_{r+1}} : \overset{k}{\Lambda}_r \to \overset{k}{\Lambda}_{r+1} \,, \qquad i_v \equiv i_{\vartheta_{r+1}} : \overset{k}{\Lambda}_r \to \overset{k}{\Lambda}_{r+1} \,,$$

which make sense by taking into account the natural inclusions $J_r\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T^*\boldsymbol{X} \subset T^*J_r\boldsymbol{Y}$ and $VJ_r\boldsymbol{Y} \subset TJ_r\boldsymbol{Y}$.

The fibred splitting (5) yields a fundamental sheaf splitting.

Lemma 1.1. *We have the splitting*

$$\overset{1}{\mathcal{H}}_{(r+1,r)} = \overset{1}{\mathcal{H}}_{r+1} \oplus \overset{1}{\mathcal{C}}_{(r+1,r)} \,,$$

where the projection on the first factor and on the second factor are given, respectively, by

$$H : \overset{1}{\mathcal{H}}_{(r+1,r)} \to \overset{1}{\mathcal{H}}_{r+1} : \alpha \mapsto i_h \alpha \,, \qquad V : \overset{1}{\mathcal{H}}_{(r+1,r)} \to \overset{1}{\mathcal{C}}_{(r+1,r)} : \alpha \mapsto i_v \alpha \,.$$

If $\alpha \in \overset{1}{\mathcal{H}}_{(r+1,r)}$ has the coordinate expression $\alpha = \alpha_\lambda d^\lambda + \alpha^{\underline{p}}_i d^i_{\underline{p}}$ $(0 \le \underline{p} \le r)$, then

$$H(\alpha) = (\alpha_\lambda + y^i_{\underline{p}} \alpha^{\underline{p}}_i)\, d^\lambda \,, \qquad V(\alpha) = \alpha^{\underline{p}}_i \vartheta^i_{\underline{p}} \,.$$

Proposition 1.1. *The above splitting of* $\overset{1}{\mathcal{H}}_{(r+1,r)}$ *induces the splitting*

$$\overset{k}{\mathcal{H}}_{(r+1,r)} = \bigoplus_{l=0}^{k} \overset{k-l}{\mathcal{C}}_{(r+1,r)} \wedge \overset{l}{\mathcal{H}}_{r+1}$$

(see Appendix A).

We recall that, in the above splitting, direct summands with $l > n$ vanish.

We set H to be the projection of the above splitting on the summand with the highest degree of the horizontal factor.

Proposition 1.2. *If $k \le n$, then we have*

$$H : \overset{k}{\mathcal{H}}_{(r+1,r)} \to \overset{k}{\mathcal{H}}_{r+1} : \alpha \mapsto \frac{1}{k!} \square^k д_{r+1}(\alpha)\,;$$

if $k > n$, then we have

$$H : \overset{k}{\mathcal{H}}_{(r+1,r)} \to \overset{k-n}{\mathcal{C}}_{(r+1,r)} \wedge \overset{n}{\mathcal{H}}_{r+1} : \alpha \mapsto \frac{1}{(k-n)!\, n!} \left(\square^{k-n} \vartheta_{r+1} \square^n д_{r+1} \right)(\alpha)\,.$$

Proof. See Appendix A. $\boxed{QED}$

We set also

$$V := Id - H$$

to be the projection complementary to H.

Remark 1.2. If $k \le n$, then we have the coordinate expression

$$H(\alpha) = y^{i_1}_{\underline{p}_1 + \lambda_1} \dots y^{i_h}_{\underline{p}_h + \lambda_h} \alpha^{\underline{p}_1 \dots \underline{p}_h}_{i_1 \dots i_h}{}_{\lambda_{h+1} \dots \lambda_k} d^{\lambda_1} \wedge \dots \wedge d^{\lambda_k} \,,$$

with $0 \le h \le k$. If $k > n$, then we have

$$\begin{aligned} H(\alpha) = \sum y^{j_1}_{\underline{q}_1 + \lambda_1} \dots y^{j_l}_{\underline{q}_l + \lambda_l} \alpha^{\underline{p}_1 \widehat{\dots} \underline{p}_{k-n+l} \underline{q}_1 \dots \underline{q}_l}_{i_1 \widehat{\dots}\, i_{k-n+l}\, j_1 \dots j_l}{}_{\lambda_{l+1} \dots \lambda_n} \\ \vartheta^{i_1}_{\underline{p}_1} \wedge \widehat{\dots} \wedge \vartheta^{i_{k-n+l}}_{\underline{p}_{k-n+l}} \wedge d^{\lambda_1} \wedge \dots \wedge d^{\lambda_n} \,, \end{aligned}$$

where $0 \le l \le n$ and the sum is over the subsets

$$\{{}^{j_1}_{\underline{q}_1} \dots {}^{j_l}_{\underline{q}_l}\} \subset \{{}^{i_1}_{\underline{p}_1} \dots {}^{i_{k-n+l}}_{\underline{p}_{k-n+l}}\} \,,$$

and $\widehat{\dots}$ stands for suppressed indexes (and corresponding contact forms) belonging to one of the above subsets. $\square$

Now, we apply the conclusion of remark 8.1 of the Appendix A to the subsheaf $\overset{k}{\Lambda}_r \subset \overset{k}{\mathcal{H}}_{(r+1,r)}$. To this aim, we want to find the image of $\overset{k}{\Lambda}_r$ under the projections of the above splitting.

We denote the restrictions of H, V to $\overset{k}{\Lambda}_r$ by h, v. Next theorem is devoted to a characterisation of the image of $\overset{k}{\Lambda}_r$ under h.

Theorem 1.1. *Let* $0 < k \leq n$, *and denote*

$$\overset{k}{\mathcal{H}}{}^h_{r+1} := h(\overset{k}{\Lambda}_r)\,.$$

Then, we have the inclusion $\overset{k}{\mathcal{H}}{}^h_{r+1} \subset \overset{k}{\mathcal{H}}{}^P_{r+1}$.

Moreover, the sheaf $\overset{k}{\mathcal{H}}{}^h_{r+1}$ *admits the following characterisation: a section* $\alpha \in \overset{k}{\mathcal{H}}{}^P_{r+1}$ *is a section of the subsheaf* $\overset{k}{\mathcal{H}}{}^h_{r+1}$ *if and only if there exists a section* $\beta \in \overset{k}{\Lambda}_r$ *such that*

$$(j_r s)^* \beta = (j_{r+1} s)^* \alpha$$

for each section $s : \boldsymbol{X} \to \boldsymbol{Y}$.

Proof. If $s : \boldsymbol{X} \to \boldsymbol{Y}$ is a section, then the following identities

$$(j_r s)^* \beta = (j_{r+1} s)^* h(\beta)\,, \qquad (j_{r+1} s)^* v(\beta) = 0\,,$$

yield

$$\alpha = h(\beta) \quad \Leftrightarrow \quad (j_r s)^* \beta = (j_{r+1} s)^* \alpha$$

for all $\alpha \in \overset{k}{\mathcal{H}}{}^P_{r+1}$ and $\beta \in \overset{k}{\Lambda}_r$. $\boxed{QED}$

Remark 1.3. It comes from the above Theorem that not any section of $\overset{k}{\mathcal{H}}{}^P_{r+1}$ is a section of $\overset{k}{\mathcal{H}}{}^h_{r+1}$; indeed, a section of $\overset{k}{\mathcal{H}}{}^P_{r+1}$ in general contains 'too many monomials' with respect to a section of $\overset{k}{\mathcal{H}}{}^h_{r+1}$. This can be seen by means of the following example. Consider a one–form $\beta \in \overset{1}{\Lambda}_0$. Then we have the coordinate expressions

$$\beta = \beta_\lambda d^\lambda + \beta_i d^i\,, \qquad h(\beta) = (\beta_\lambda + y^i_\lambda \beta_i) d^\lambda\,.$$

If $\alpha \in \overset{1}{\mathcal{H}}{}^P_1$, then we have the coordinate expression

$$\alpha = (\alpha_\lambda + y^j_\mu \alpha_j^\mu{}_\lambda) d^\lambda\,.$$

It is evident that, in general, there does not exists $\beta \in \overset{1}{\Lambda}_r$ such that $h(\beta) = \alpha$. $\square$

Corollary 1.1. *Let* $\dim \boldsymbol{X} = 1$*. Then we have*

$$\overset{1}{\mathcal{H}}{}^{h}_{r+1} = \overset{1}{\mathcal{H}}{}^{P}_{r+1} \,.$$

PROOF. From the above coordinate expressions. See also [26]. QED

Lemma 1.2. *The sheaf morphisms* H, V *restrict on the sheaf* $\overset{k}{\Lambda}_r$ *to the surjective sheaf morphisms*

$$h : \overset{1}{\Lambda}_r \to \overset{1}{\mathcal{H}}{}^{h}_{r+1} \,, \qquad v : \overset{1}{\Lambda}_r \to \overset{1}{\mathcal{C}}_r \,.$$

PROOF. The restriction of H has already been studied. As for the restriction of V, it is easy to see by means of a partition of the unity that it is surjective on $\overset{1}{\mathcal{C}}_r$. QED

Theorem 1.2. *The splitting of proposition 1.1 yields the inclusion*

$$\overset{k}{\Lambda}_r \subset \bigoplus_{l=0}^{k} \overset{k-l}{\mathcal{C}}{}_r \wedge \overset{l}{\mathcal{H}}{}^{h}_{r+1} \,,$$

and the splitting projections restrict to surjective maps.

PROOF. In fact, for any $l \leq k$ the restriction of the projection

$$\overset{k}{\mathcal{H}}_{(r+1,r)} \to \overset{k-l}{\mathcal{C}}{}_{(r+1,r)} \wedge \overset{l}{\mathcal{H}}_{r+1}$$

of the splitting of proposition 1.1 to the sheaf $\overset{k}{\Lambda}_r$ takes the form

$$\overset{k}{\Lambda}_r \to \overset{k-l}{\mathcal{C}}{}_r \wedge \overset{l}{\mathcal{H}}{}^{h}_{r+1} \subset \overset{k-l}{\mathcal{C}}{}_{(r+1,r)} \wedge \overset{l}{\mathcal{H}}_{r+1} \,.$$

The above inclusion can be tested in coordinates. For the sake of simplicity, let us consider a global section $\alpha \in \overset{k-l}{\mathcal{C}}{}_r \wedge \overset{l}{\mathcal{H}}{}^{h}_{r+1}$ where $0 \leq l \leq n$. We have the coordinate expression

$$\alpha = y^{j_1}_{\underline{q}_1+\lambda_1} \dots y^{j_h}_{\underline{q}_h+\lambda_h} \alpha^{\underline{p}_1 \dots \underline{p}_{k-l} \underline{q}_1 \dots \underline{q}_h}_{i_1 \dots i_{k-l}\, j_1 \dots j_h \lambda_{h+1} \dots \lambda_l} \\ \vartheta^{i_1}_{\underline{p}_1} \wedge \dots \wedge \vartheta^{i_{k-l}}_{\underline{p}_{k-l}} \wedge d^{\lambda_1} \wedge \dots \wedge d^{\lambda_l} \,,$$

where $0 \leq |\underline{p}_i|, |\underline{q}_i| \leq r$ and $0 \leq h \leq n$. If $\{\psi_i\}$ is a partition of the unity on $\overset{0}{\Lambda}_r$ subordinate to a coordinate atlas, let

$$\tilde{\alpha}_i := \psi_i \, \tilde{\alpha}^{\underline{s}_1 \dots \underline{s}_r}_{t_1 \dots t_r \lambda_{r+1} \dots \lambda_k} d^{t_1}_{\underline{s}_1} \wedge \dots \wedge d^{t_r}_{\underline{p}_r} \wedge d^{\lambda_{r+1}} \wedge \dots \wedge d^{\lambda_k} \,,$$

where the set of pairs of indices $\{{}^{t_1}_{\underline{s}_1} \dots {}^{t_r}_{\underline{s}_r}\}$ is a permutation of the set of pairs of indices $\{{}^{i_1}_{\underline{p}_1} \dots {}^{i_{k-l}}_{\underline{p}_{k-l}} {}^{j_1}_{\underline{q}_1} \dots {}^{j_l}_{\underline{q}_l}\}$. Then $\sum_i \tilde{\alpha}_i$ is a global section of $\overset{k}{\Lambda}_r$ whose projection on $\overset{k-l}{\mathcal{C}}{}_r \wedge \overset{l}{\mathcal{H}}{}^{h}_{r+1}$ is α.

The proof is analogous for $k > n$. QED

We remark that, in general, the above inclusion is a proper inclusion: in general, a sum of elements of the direct summands is not an element of $\overset{k}{\Lambda}_r$.

Corollary 1.2. *The sheaf morphism H restricts on the sheaf $\overset{k}{\Lambda}_r$ to the surjective sheaf morphisms*

$$h : \overset{k}{\Lambda}_r \to \overset{k}{\mathcal{H}}{}^h_{r+1} \qquad k \leq n\,, \qquad h : \overset{k}{\Lambda}_r \to \overset{k-n}{\mathcal{C}}{}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1} \qquad k > n\,.$$

Horizontal and vertical differential

The derivations i_h, i_v, and the exterior differential d yield two derivations of degree one (see [17]). Namely, we define the *horizontal* and *vertical differential* to be the sheaf morphisms

$$d_h := i_h \circ d - d \circ i_h : \overset{k}{\Lambda}_r \to \overset{k}{\Lambda}_{r+1}\,, \qquad d_v := i_v \circ d - d \circ i_v : \overset{k}{\Lambda}_r \to \overset{k}{\Lambda}_{r+1}\,,$$

It can be proved (see [17]) that d_h and d_v fulfill the properties

$$\begin{gathered} d_h^2 = d_v^2 = 0\ , \qquad d_h \circ d_v + d_v \circ d_h = 0\ , \\ d_h + d_v = (\pi^{r+1}_r)^* \circ d\ , \\ (j_{r+1}s)^* \circ d_v = 0\ , \qquad d \circ (j_r s)^* = (j_{r+1}s)^* \circ d_h\ . \end{gathered}$$

The action of d_h and d_v on functions $f : J_r\boldsymbol{Y} \to I\!\!R$ and one–forms on $J_r\boldsymbol{Y}$ uniquely characterises d_h and d_v. We have the coordinate expressions

$$\begin{gathered} d_h f = (\text{д}_{r+1})_\lambda.f d^\lambda = (\partial_\lambda f + y^i_{\underline{p}+\lambda} \partial^{\underline{p}}_i f) d^\lambda\,, \\ d_h d^\lambda = 0\,, \qquad d_h d^i_{\underline{p}} = -d^i_{\underline{p}+\lambda} \wedge d^\lambda\,, \qquad d_h \vartheta^i_{\underline{p}} = -\vartheta^i_{\underline{p}+\lambda} \wedge d^\lambda\,, \\ d_v f = \partial^{\underline{p}}_i f \vartheta^i_{\underline{p}}\ , \\ d_v d^\lambda = 0\,, \qquad d_v d^i_{\underline{p}} = d^i_{\underline{p}+\lambda} \wedge d^\lambda\,, \qquad d_v \vartheta^i_{\underline{p}} = 0\,. \end{gathered}$$

We note that

$$-d^i_{\underline{p}+\lambda} \wedge d^\lambda = -\vartheta^i_{\underline{p}+\lambda} \wedge d^\lambda + y^i_{\underline{p}+\lambda+\mu} d^\mu \wedge d^\lambda = -\vartheta^i_{\underline{p}+\lambda} \wedge d^\lambda\,.$$

Finally, next Proposition analyses the relationship of d_h and d_v with the splitting of Proposition 1.1.

Proposition 1.3. *We have*

$$\begin{gathered} d_h\left(\overset{k}{\mathcal{H}}_r\right) \subset \overset{k+1}{\mathcal{H}}{}_{r+1}\,, \qquad d_v\left(\overset{k}{\mathcal{H}}_r\right) \subset \overset{1}{\mathcal{C}}_r \wedge \overset{k}{\mathcal{H}}_r\,, \\ d_h\left(\overset{k}{\mathcal{C}}_{(r,r-1)} \wedge \overset{h}{\mathcal{H}}_r\right) \subset \overset{k}{\mathcal{C}}_{(r+1,r)} \wedge \overset{h+1}{\mathcal{H}}{}_{r+1}\,, \qquad d_h\left(\overset{k}{\mathcal{C}}_{(r,r-1)} \wedge \overset{n}{\mathcal{H}}_r\right) = \{0\}\,, \\ d_v\left(\overset{k}{\mathcal{C}}_{(r,r-1)}\right) \subset \overset{k+1}{\mathcal{C}}{}_r\,, \qquad d_v\left(\overset{k}{\mathcal{C}}_r\right) \subset \overset{k+1}{\mathcal{C}}{}_r\,, \end{gathered}$$

Proof. From the action of d_h, d_v on functions and local coordinate bases of forms.

QED

Direct limit

The sheaf injections π^r_s $(r \geq s)$ provide several inclusions between the sheaves of forms previously introduced. This yields several injective systems, whose direct limit is studied here.

We define the presheaves on $\boldsymbol{Y}$

$$\overset{k}{\Lambda} := \lim_{\rightarrow} \overset{k}{\Lambda}_r \,, \qquad \overset{(k,h)}{\Lambda} := \lim_{\rightarrow} \overset{(k,h)}{\Lambda}_{(r+1,r)} \,.$$

By simple counterexamples, it can be proved that the above presheaves are not sheaves in general, because the gluing axiom fails to be true.

Remark 1.4. For any equivalence class $[\alpha] \in \overset{k}{\Lambda}$ there exists a distinguished representative $\beta \in \overset{k}{\Lambda}_r$ whose order r is minimal. The same holds for $\overset{(0,k)}{\Lambda}$ and $\overset{(k,0)}{\Lambda}$. Accordingly, we shall often indicate by $\beta \in \overset{k}{\Lambda}$ (without brackets) such a minimal section. □

Lemma 1.3. *We have* $\lim\limits_{\rightarrow} \overset{k}{\Lambda}_{(r+1,r)} = \lim\limits_{\rightarrow} \overset{k}{\Lambda}_r \equiv \overset{k}{\Lambda}$.

PROOF. In fact, we have the inclusions $\overset{k}{\Lambda}_r \subset \overset{k}{\Lambda}_{(r+1,r)} \subset \overset{k}{\Lambda}_{r+1}$ QED

Theorem 1.3. *We have the natural splitting*

$$\overset{k}{\Lambda} = \bigoplus_{l=0}^{k} \overset{(k-l,l)}{\Lambda} \,.$$

PROOF. It comes from the above lemma and the splitting of proposition 1.1. QED

Remark 1.5. The above splitting represents one of the major differencies between the finite order and the infinite order case. As we shall see, in the infinite order formulations one has to deal with quotients of $\overset{k}{\Lambda}$ by sheaves of contact forms. The above splitting allows us to identify such quotients with 'more concrete' spaces (see proposition 4.2). The situation is much more complicated in the finite order case for the lack of such a splitting. In fact, the inclusion $\overset{k}{\Lambda}_r \subset \overset{k}{\Lambda}_{(r+1,r)}$ is a proper inclusion, and we are in the bad situation described in remark 8.1. Nevertheless, by means of the splitting of proposition 1.1, we are able to recover in the finite order case almost all features of infinite order formulations, but in a much more difficult way (see [28]). □

Proposition 1.4. *The sheaf morphisms d, d_h, d_v, $\overset{k}{h}$, admit direct limits. Namely, such* *direct limits turn out to be the presheaf morphisms*

$$d : \overset{k}{\Lambda} \to \overset{k+1}{\Lambda} : [\alpha] \mapsto [d\alpha]\,,$$
$$d_h : \overset{k}{\Lambda} \to \overset{k+1}{\Lambda} : [\alpha] \mapsto [d_h\alpha]\,, \quad d_v : \overset{k}{\Lambda} \to \overset{k+1}{\Lambda} : [\alpha] \mapsto [d_v\alpha]\,,$$
$$\overset{k}{h} : \overset{k}{\Lambda}_{(r+1,r)} \to \begin{cases} \overset{(0,n)}{\Lambda}_{r+1} : [\alpha] \mapsto [\overset{k}{h}] & \textit{if } k \le n \\ \overset{(k-n,n)}{\Lambda}_{r+1} : [\alpha] \mapsto [\overset{k}{h}] & \textit{if } k > n\,; \end{cases}$$

Note that the map $\overset{k}{h}$ of the above proposition turns out to be the projection of the splitting of theorem 1.3 on the factor with the highest horizontal degree; in other words, the direct limit of the projection is the projection of the splitting of the direct limit.

We observe that we did not indicate the degree of d, d_h and d_v. This is both for a matter of 'tradition' and not to make too heavy the notation.

Finally, next proposition analyses the relationship of d_h and d_v with the splitting of the above theorem.

Proposition 1.5. *We have*

$$d_h(\overset{(0,k)}{\Lambda}) \subset \overset{(0,k+1)}{\Lambda}\,, \qquad d_v(\overset{(0,k)}{\Lambda}) \subset \overset{(1,k)}{\Lambda}\,,$$
$$d_h(\overset{(k,0)}{\Lambda}) \subset \overset{(k,1)}{\Lambda}\,, \qquad d_v(\overset{(k,0)}{\Lambda}) \subset \overset{(k+1,0)}{\Lambda}\,.$$

PROOF. From the action of d_h, d_v on functions and local coordinate bases of forms. $\boxed{QED}$

2 Finite order variational sequence

In this section, we recall the theory of variational sequences on finite order jet bundles [10]. We give a concise summary of the theory using our notation.

We consider the de Rham exact sheaf sequence on $J_r\boldsymbol{Y}$

$$0 \longrightarrow I\!R \longrightarrow \overset{0}{\Lambda}_r \xrightarrow{d} \overset{1}{\Lambda}_r \xrightarrow{d} \dots \xrightarrow{d} \overset{J}{\Lambda}_r \xrightarrow{d} 0,$$

where $J := \dim J_r\boldsymbol{Y}$. We are able to provide several natural subsequences of the de Rham sequence. For example, natural subsequences of the de Rham sequence arise by considering the ideals generated in $\overset{k}{\Lambda}_r$ by its natural subsheaves $\overset{1}{\mathcal{H}}_{(r,s)}$, $\overset{1}{\mathcal{C}}_{(r,s)}$, ... Not all natural subsequences of the de Rham sequence turn out to be exact. Here, we introduce an exact natural subsequence of the de Rham sequence, which is of particular importance in the variational calculus, although being defined independently (see [10, 26]).

We introduce a new subsheaf of $\overset{k}{\Lambda}_r$. Namely, we set

$$\mathcal{C}\overset{k}{\Lambda}_r = \{\alpha \in \overset{k}{\Lambda}_r \mid (j_r s)^*\alpha = 0 \text{ for every section } s : \boldsymbol{X} \to \boldsymbol{Y}\} .$$

The above subsheaf $\mathcal{C}\overset{n}{\Lambda}_r$ is made by forms which does not give contribution to action–like functionals. [10, 17, 27].

Lemma 2.1. *We have*

$$\mathcal{C}\overset{k}{\Lambda}_r = \ker h \quad \textit{if} \quad 0 \le k \le n\,, \qquad \mathcal{C}\overset{k}{\Lambda}_r = \overset{k}{\Lambda}_r \quad \textit{if} \quad k > n\,.$$

Proof. Let $\alpha \in \overset{k}{\Lambda}_r$. Then, for any section $s : \boldsymbol{X} \to \boldsymbol{Y}$ we have

$$(j_r s)^*\alpha = (j_{r+1}s)^* h(\alpha)\,,$$

and $\alpha \in \ker h$ implies $\alpha \in \mathcal{C}\overset{k}{\Lambda}_r$. Conversely, suppose $\alpha \in \mathcal{C}\overset{k}{\Lambda}_r$. Then we have

$$(j_{r+1}s)^* h(\alpha) = h(\alpha)_{\lambda_1 \dots \lambda_k} \circ j_{r+1}s \;\; d^\lambda_1 \wedge \dots \wedge d^\lambda_k\,,$$

hence $h(\alpha) = 0$.

The first assertion comes from the above identities and $\dim \boldsymbol{X} = n$. $\boxed{\scriptstyle QED}$

We set $\overset{k}{\Theta}_r$ to be the sheaf generated (in the sense of [29]) by the presheaf $\ker h + d \ker h$.

Remark 2.1. We stress that, in general, the sheaf axioms fail to be true for $d \ker h$. Anyway, if $\dim \boldsymbol{X} = 1$ and $k > 1$, the sum $\ker h + d\ker h$ turns out to be a direct sum [26], and $d\ker h$ turns out to be a sheaf. $\square$

In the rest of this section, we also denote by $d \ker h$ the sheaf generated by the presheaf $d \ker h$, by an abuse of notation.

Lemma 2.2. *If* $0 \le k \le n$, *then* $d\ker h \subset \ker h$, *so that* $\overset{k}{\Theta}_r = \mathcal{C}\overset{k}{\Lambda}_r$.

Proof. By the above Lemma, if $\alpha \in \ker h$, then for any section $s : \boldsymbol{X} \to \boldsymbol{Y}$ we have $(j_r s)^*\alpha = 0$, hence $(j_r s)^* d\alpha = 0$. So, $d\alpha \in \ker h$. $\boxed{\scriptstyle QED}$

It is clear that $\overset{k}{\Theta}_r$ is a subsheaf of $\overset{k}{\Lambda}_r$. Thus, we say the following natural subsequence

$$0 \longrightarrow \overset{1}{\Theta}_r \xrightarrow{\;d\;} \overset{2}{\Theta}_r \xrightarrow{\;d\;} \dots \xrightarrow{\;d\;} \overset{I}{\Theta}_r \xrightarrow{\;d\;} 0$$

to be the *contact subsequence* of the de Rham sequence. We note that, in general, the sheaves $\overset{k}{\Theta}_r$ are not the sheaves of sections of a vector subbundle of $T^* J_r \boldsymbol{Y}$.

Remark 2.2. In general, I depends on the dimension of the fibers of $J_r\boldsymbol{Y} \to \boldsymbol{X}$; its value is given in [10]. $\square$

The following theorem is proved in [10].

Theorem 2.1. *The contact subsequence is exact and soft.*

Now, we introduce a bicomplex by quotienting the de Rham sequence on $J_r\boldsymbol{Y}$ by the contact subsequence. We obtain a new sequence, the variational sequence, which turns out to be exact. In the last part of the section, we describe the relationships between bicomplexes on jet spaces of different orders.

Proposition 2.1. *The following diagram*

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \overset{1}{\Theta}_r \arrow[r, "d"] \arrow[d] & \overset{2}{\Theta}_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & \overset{I}{\Theta}_r \arrow[r, "d"] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d] & \overset{0}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \overset{1}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \overset{2}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & \overset{I}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \overset{I+1}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d] & \overset{0}{\Lambda}_r \arrow[r, "\mathcal{E}_0"] \arrow[d] & \overset{1}{\Lambda}_r/\overset{1}{\Theta}_r \arrow[r, "\mathcal{E}_1"] \arrow[d] & \overset{2}{\Lambda}_r/\overset{2}{\Theta}_r \arrow[r, "\mathcal{E}_2"] \arrow[d] & \dots \arrow[r, "\mathcal{E}_{I-1}"] & \overset{I}{\Lambda}_r/\overset{I}{\Theta}_r \arrow[r, "\mathcal{E}_I"] \arrow[d] & \overset{I+1}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & 0 \\
 & 0 & 0 & 0 & 0 & & 0 & 0 & &
\end{tikzcd}
$$

is a commutative diagram whose rows and columns are exact.

Proof. We have to prove only the exactness of the bottom row of the diagram. But this follows from the exactness of the other rows and of the columns.

Definition 2.1. We say the bottom row of the above diagram to be the r–th order *variational sequence* associated with the fibred manifold $\boldsymbol{Y} \to \boldsymbol{X}$ (see [10]). □

We stress that this sequence is obtained in an intrinsic way, but it is not the unique intrinsic one. It is obtained in order to match precise criteria, i.e. to obtain an exact sheaf sequence that carries the appropriate information for the calculus of variations.

Proposition 2.2. *The sheaves $\overset{k}{\Lambda}_r/\overset{k}{\Theta}_r$ are soft sheaves [10].*

Proof. In fact, each column is a short exact sheaf sequence in which $\overset{k}{\Theta}_r$ and $\overset{k}{\Lambda}_r$ are soft sheaves (see [29]). $\boxed{QED}$

Corollary 2.1. *The variational sequence is a soft resolution of the constant sheaf $\mathbb{R}$ over $\boldsymbol{Y}$ [10].*

Proof. In fact, except $I\!R$, each one of the sheaves in the sequence is soft [29].

The most interesting consequence of the above corollary is the following one (for a proof, see [29]). Let us consider the cochain complex

$$0 \longrightarrow I\!R_{\boldsymbol{Y}} \longrightarrow \left(\overset{0}{\Lambda}_r\right)_{\boldsymbol{Y}} \xrightarrow{d} \left(\overset{1}{\Lambda}_r/\overset{1}{\Theta}_r\right)_{\boldsymbol{Y}} \xrightarrow{\mathcal{E}_1} \dots \xrightarrow{d} \left(\overset{J}{\Lambda}_r\right)_{\boldsymbol{Y}} \xrightarrow{d} 0$$

and denote by H^k_{VS} the k–th cohomology group of the above cochain complex.

Corollary 2.2. *For all $k \geq 0$ there is a natural isomorphism*

$$H^k_{VS} \simeq H^k_{de\ Rham}\boldsymbol{Y}$$

(see [10]).

Proof. In fact, the Lagrangian sequence is a soft resolution of $I\!R$, hence the cohomology of the sheaf $I\!R$ is naturally isomorphic to the cohomology of the above cochain complex. Also, the de Rham sequence gives rise to a cochain complex of global sections, whose cohomology is naturally isomorphic to the cohomology of the sheaf $I\!R$ on $\boldsymbol{Y}$. Hence, we have the result by a composition of isomorphisms. (See [29] for more details on the above natural isomorphisms.) $\boxed{QED}$

3 Finite order $\mathcal{C}$–spectral sequence

The $\mathcal{C}$–spectral sequence has been introduced by Vinogradov [23, 24, 25]. It is a very powerful tool in the study of differential equations.

Here, we present a new finite order approach to variational sequences by means of the $\mathcal{C}$–spectral sequence induced by the de Rham exact sequence $(\overset{*}{\Lambda}_r, d)$ (see Lemma 8.1) on the jet space of order r of a fibred manifold. It shall be remarked that such an approach has already been attempted in a very particular case [5]. Indeed, our finite order formulation presents some technical difficulties: our main tool is the splitting of Theorem 1.2, where the direct summands have a rather complicated structure and, above all, are not subsheaves of $\overset{k}{\Lambda}_r$.

Then, we show the correspondence between the simplified finite order variational sequence and the variational sequence obtained via the finite order $\mathcal{C}$–spectral sequence.

Remark 3.1. The finite order $\mathcal{C}$–spectral sequence is formulated here in the category of presheaves of $I\!R$–vector spaces. This means that the constructions of Appendix B will be done on any open set. We stress that the reason for doing this a lie in the fact that, in our case, the function mapping open sets into homology groups is not a sheaf, but just a presheaf. $\square$

We consider the sheaf of differential groups $(\overset{*}{\Lambda}_r, d)$ and the graded sheaf filtration $(\mathcal{C}^p\overset{*}{\Lambda}_r, d)_{p\in\mathbb{N}}$, where

$$\mathcal{C}^1\overset{*}{\Lambda}_r := \mathcal{C}\overset{*}{\Lambda}_r \equiv \{\alpha \in \overset{*}{\Lambda}_r \mid \forall\, s \,\text{section of}\, \boldsymbol{Y} \to \boldsymbol{X}\ (j_r s)^*\alpha = 0\}$$

and $\mathcal{C}^p\overset{*}{\Lambda}_r$ is the p–th power of the ideal $\mathcal{C}^1\overset{*}{\Lambda}_r$ in $\overset{*}{\Lambda}_r$. We set $\mathcal{C}^0\overset{*}{\Lambda}_r = \overset{*}{\Lambda}_r$, and $\mathcal{C}^p\overset{k}{\Lambda}_r = \{0\}$ if $p > k$. We recall that

$$E_0^{p,q} \equiv \mathcal{C}^p\overset{p+q}{\Lambda}{}_r / \mathcal{C}^{p+1}\overset{p+q}{\Lambda}{}_r\,.$$

Moreover, we recall the exact sequence of Lemma 8.3.

As a preliminar step, we look for a description of the sheaves $\mathcal{C}^p\overset{p+q}{\Lambda}{}_r$. To this aim, we introduce new projections associated to the splitting of proposition 1.1

Let $0 \le q \le n$; we denote by H^p the projection

$$\overset{p+q}{\mathcal{H}}{}_{(r+1,r)} \to \bigoplus_{l=1}^{p} \overset{p-l}{\mathcal{C}}{}_{(r+1,r)} \wedge \overset{q+l}{\mathcal{H}}{}_{r+1}\,;$$

we denote by V^p the complementary projection, i.e. $V^p = \mathrm{id} - H^p$. Of course, $H^p = 0$ if $q = n$. Also, we denote by h^p and v^p the corresponding restrictions to the subsheaf $\overset{k}{\Lambda}_r$.

Lemma 3.1. *We have*

$$H^1 = H\,, \qquad\qquad H^p = H \quad \textit{if} \quad q = n-1\,.$$

Remark 3.2. By the above lemma, if $p > 1$ and $q < n-1$ then h^p is not surjective on $\oplus_{l=1}^{p} \overset{p-l}{\mathcal{C}}{}_r \wedge \overset{q+l}{\mathcal{H}}{}^h_{r+1}$, in general. But the most interesting cases are $p = 1$ and $q = n-1$, where $h^p = h$ is surjective.

Lemma 3.2. *Let $p \ge 1$. Then, we have*

$$\mathcal{C}^p\overset{p+q}{\Lambda}{}_r \simeq \ker h^p \quad \textit{if} \quad q < n\,;$$
$$\mathcal{C}^p\overset{p+q}{\Lambda}{}_r = \overset{p+q}{\Lambda}{}_r \quad \textit{if} \quad q \ge n\,.$$

PROOF. We recall that (lemma 2.1 and lemma 3.1) the theorem holds for $p = 1$. Then, we have the identities $\ker H^p = \operatorname{im} V^p$ and $\operatorname{im} V^p = \langle(\operatorname{im} V)^p\rangle = \langle(\ker H)^p\rangle$, where $\langle(\operatorname{im} V)^p\rangle$ denotes the ideal generated by pth exterior powers of elements of $\operatorname{im} V$ in $\overset{1+q}{\Lambda}{}_r$. So, by restriction to $\overset{k}{\Lambda}_r$, we have $\ker h^p = \langle(\ker h)^p\rangle$. But, by definition and lemma 2.1, we have $\mathcal{C}^p\overset{p+q}{\Lambda}{}_r = \langle(\ker h)^p\rangle$, hence the result. $\boxed{QED}$

Now, we compute (E_0, e_0).

Lemma 3.3. *We have*

$$
\begin{aligned}
&E_0^{p,0} = \ker h^p\,;\\
&E_0^{p,q} \subset \overset{p}{\mathcal{C}}_r \wedge \overset{q}{\mathcal{H}}{}^h_{r+1} \quad \textit{if} \quad 1 \leq q < n\,;\\
&E_0^{p,n} \simeq \overset{p}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}\\
&E_0^{p,q} = \{0\} \quad \textit{otherwise;}\\
&\bar{d} \equiv e_0^{p,q} : E_0^{p,q} \to E_0^{p,q+1} : h^{p+1}(\alpha) \mapsto h^{p+2}(d\alpha)\,.
\end{aligned}
$$

Proof. The first and fourth assertions are trivial. As for the second one, the inclusion is realised via the injective morphism

$$E_0^{p,q} \equiv \ker h^p \big/ \ker h^{p+1} \to \overset{p}{\mathcal{C}}_r \wedge \overset{q}{\mathcal{H}}{}^h_{r+1} : [\alpha] \mapsto h^{p+1}(\alpha)\,.$$

The third statement comes from the identity $h^p = 0$ if $q = n$, which imply $\ker h^p = \overset{p+n}{\Lambda}$, and lemma 3.1, which imply that h^{p+1} is surjective.

The sheaf morphism $\bar{d}$ can be read through the above morphism; we obtain the last assertion. $\boxed{QED}$

Proposition 3.1. *The bigraded complex (E_0, e_0) is isomorphic to the sequence of cochain complexes*

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \dots & 0 \arrow[d] \\
\overset{0}{\Lambda}_r \arrow[d, "\bar{d}"'] & \ker h^1 \arrow[d, "-\bar{d}"'] & \ker h^2 \arrow[d, "\bar{d}"'] & \dots & \ker h^I \arrow[d, "(-1)^I\bar{d}"'] \\
\overset{1}{\mathcal{H}}{}^h_r \arrow[d, "\bar{d}"'] & E_0^{1,1} \arrow[d, "-\bar{d}"'] & E_0^{2,1} \arrow[d, "\bar{d}"'] & \dots & E_0^{I,1} \arrow[d, "(-1)^I\bar{d}"'] \\
\dots \arrow[d, "\bar{d}"'] & \dots \arrow[d, "-\bar{d}"'] & \dots \arrow[d, "\bar{d}"'] & \dots & \dots \arrow[d, "(-1)^I\bar{d}"'] \\
\overset{n}{\mathcal{H}}{}^h_r \arrow[d, "\bar{d}"'] & \overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1} \arrow[d, "-\bar{d}"'] & \overset{2}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1} \arrow[d, "\bar{d}"'] & \dots & \overset{I}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1} \arrow[d, "(-1)^I\bar{d}"'] \\
0 & 0 & 0 & & 0
\end{tikzcd}
$$

The sequence becomes trivial after the I–th column.

The minus signs are put in order to agree with an analogous convention on infinite order variational bicomplexes.

Remark 3.3. As we will see, in the infinite order case the sheaf morphism d_v yields horizontal arrows in the sequence analogous to the above one. So, we obtain a commutative bicomplex. Here, we have no horizontal arrows, due to the fact that the maps d_v raise the order of the jet by one.

Another difference with the infinite order sequence is that here the sequence becomes trivial after a certain value of the degree p. □

We note that the bottom row of the above sequence projects to 0. Also, we recall that $E_1 = H(E_0)$, where the homology is taken with respect to the sheaf morphism d_h. These two facts yield the following Theorem.

Theorem 3.1. *We have the bicomplex*

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & \overset{0}{\Lambda}_r \arrow[d, "\bar{d}"'] & \dots & \ker h^p \arrow[d, "(-1)^I \bar{d}"'] & \dots \\
 & \overset{1}{\mathcal{H}}{}^h_r \arrow[d, "\bar{d}"'] & \dots & E_0^{p,1} \arrow[d, "(-1)^I \bar{d}"'] & \dots \\
 & \dots \arrow[d, "\bar{d}"'] & \dots & \dots \arrow[d, "(-1)^I \bar{d}"'] & \dots \\
 & \overset{n}{\mathcal{H}}{}^h_r \arrow[d, "\bar{d}"'] & \dots & \overset{p}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1} \arrow[d, "(-1)^I \bar{d}"'] & \dots \\
0 \arrow[r] & \overset{n}{\mathcal{H}}{}^h_r / \bar{d}(\overset{n-1}{\mathcal{H}}{}^h_r) \arrow[r, "\mathcal{E}'_n"] \arrow[d] & \dots \arrow[r, "\mathcal{E}'_{n+p-1}"] & (\overset{p}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}) / \bar{d}(E_0^{p,n-1}) \arrow[r, "\mathcal{E}'_{n+p}"] \arrow[d] & \dots \\
 & 0 & & 0 &
\end{tikzcd}$$

where the bottom row is a presheaf of cochain complexes. The bicomplex is trivial if $p > I$ and vertical arrows with values into the quotients are trivial projections. We *have the identifications*

$$
\begin{aligned}
E_1^{0,n} &= \overset{n}{\mathcal{H}}{}^h_{r+1} / \bar{d}(\overset{n-1}{\mathcal{H}}{}^h_{r+1})\,,\\
E_1^{p,n} &= (\overset{p}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}) / \bar{d}(\overset{p}{\mathcal{C}}_r \wedge \overset{n-1}{\mathcal{H}}{}^h_{r+1})\,,\\
e_1^{0,n} &= \mathcal{E}'_n : \overset{n}{\mathcal{H}}{}^h_{r+1} / \bar{d}(\overset{n-1}{\mathcal{H}}{}^h_{r+1}) \to (E_0^{1,n-1}) / \bar{d}(\overset{1}{\mathcal{C}}_r \wedge \overset{n-1}{\mathcal{H}}{}^h_{r+1}) :\\
&\qquad [h^1(\alpha)] \mapsto [h^2(d\alpha)]\,,\\
e_1^{p,n} &= \mathcal{E}'_{p+n} : (\overset{p}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}) / \bar{d}(E_0^{p,n-1}) \to (\overset{p+1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}) / \bar{d}(E_0^{p+1,n-1}) :\\
&\qquad [h^{p+1}(\alpha)] \mapsto [h^{p+2}(d\alpha)]\,.
\end{aligned}
$$

Proof. The above identifications come directly from the definition of E_1. As for the last statement, by recalling the exact sequence of Lemma 8.3, we have by definition

$$
e_1^{p,1} = \pi \circ \delta\,,
$$

where δ is the Bockstein operator induced by the exact sequence and π is the cohomology map induced by the corresponding map π of the exact sequence. So, suppose that

$$
h^{p+1}(\alpha) \in E_0^{p,n} = \overset{p}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}\,;
$$

we have $\alpha \in \overset{p+n}{\Lambda}_r$. Then,

$$
\pi(d\alpha) = \bar{d}(\pi(\alpha)) = 0\,,
$$

because $\bar{d}$ raises the degree by 1 on the horizontal factor, so, $d\alpha \in \mathcal{C}^{p+1}\overset{p+1+n}{\Lambda}_r$. Being $d(d\alpha) = 0$, $d\alpha$ is closed in $\mathcal{C}^{p+1}\overset{p+1+n}{\Lambda}_r$ under the differential d, but $d\alpha$ is not exact in $\mathcal{C}^{p+1}\overset{p+n}{\Lambda}_r$, i.e. there does not exist a form $\beta \in \mathcal{C}^{p+1}\overset{p+n}{\Lambda}_r = \ker h^{p+1}$ such that $d\beta = \alpha$. Hence, $d\alpha$ determines a cohomology class $[d\alpha]$ in $\mathcal{C}^{p+1}\overset{p+1+n}{\Lambda}_r$ which is, by definition, the value of $\delta([h^{p+1}(\alpha)])$. The map π maps $d\alpha$ into $h^{p+2}(d\alpha)$, hence the cohomology class $[d\alpha]$ is mapped into $[h^{p+2}(d\alpha)]$ by π. $\boxed{QED}$

Theorem 3.2. *We have the commutative diagram*

$$
\begin{tikzcd}
 & 0 \arrow[d] & & \\
 & \overset{0}{\Lambda}_r \arrow[d, "\bar{d}"'] & & \\
 & \overset{1}{\mathcal{H}}{}^h_r \arrow[d, "\bar{d}"'] & & \\
 & \dots \arrow[d, "\bar{d}"'] & & \\
 & \overset{n}{\mathcal{H}}{}^h_r \arrow[d] \arrow[dr, "\tilde{\mathcal{E}}_n"] & & \\
0 \arrow[r] & \overset{n}{\mathcal{H}}{}^h_r / \bar{d}(\overset{n-1}{\mathcal{H}}{}^h_r) \arrow[r, "\mathcal{E}'_n"] \arrow[d] & (\overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}) / \bar{d}(E_0^{1,n-1}) \arrow[r, "\mathcal{E}'_{n+1}"] & \dots \\
 & 0 & &
\end{tikzcd}
$$

where $\tilde{\mathcal{E}}_n := \mathcal{E}'_n \circ \bar{d}$, and the sequence

$$\dots \xrightarrow{\tilde{\mathcal{E}}_{n-1}} \overset{n}{\mathcal{H}}{}^h_r \xrightarrow{\tilde{\mathcal{E}}_n} (\overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}) / \bar{d}(E_0^{1,n-1}) \xrightarrow{\mathcal{E}'_{n+1}} \dots$$

is a complex of presheaves.

Definition 3.1. We say the bottom row of the above bicomplex to be the *finite order variational sequence associated with the finite order $\mathcal{C}$–spectral sequence on $\boldsymbol{Y} \to \boldsymbol{X}$.* □

The cohomology of the above sequence will be clear in next section after proving that it is isomorphic with the finite order variational sequence of definition 2.1.

4 Comparison between finite order approaches

In this section, we show the connection between Krupka's variational sequence and the variational sequence associated with the finite order $\mathcal{C}$–spectral sequence.

First of all, we provide a simplified version of Krupka's variational sequence , i.e. , a sequence which is isomorphic to Krupka's variational sequence but is made by sheaves of forms or by quotient sheaves which are quotients between 'smaller' sheaves.

In the case $0 \leq k \leq n$, lemma 2.2 yields immediately the following result.

Theorem 4.1. *Let $0 \leq k \leq n$. Then, the sheaf morphism h yields the isomorphism*

$$I_k : \overset{k}{\Lambda}_r / \overset{k}{\Theta}_r \to \overset{k}{\mathcal{H}}{}^h_{r+1} : [\alpha] \mapsto h(\alpha)\,.$$

In the case $k > n$, we are able to provide isomorphisms of the quotient sheaves with other quotient sheaves made with proper subsheaves.

Proposition 4.1. *Let $k > n$. Then, the projection h induces the natural sheaf isomorphism*

$$\left(\overset{k}{\Lambda}_r / \overset{k}{\Theta}_r\right) \to \left(\overset{k-n}{\mathcal{C}}{}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}\right) / h(d \ker h) : [\alpha] \mapsto [h(\alpha)]\ ,$$

where $d \ker h$ stands for the sheaf generated by the presheaf $d \ker h$, by an abuse of notation.

Proof. The map is clearly well defined.

Also, the map is injective, for if $\alpha, \alpha' \in \overset{k}{\Lambda}_r$, then

$$[h(\alpha)] = [h(\alpha')] \Rightarrow h(\alpha - \alpha') = hdp\ ,$$

with $p \in \ker h$. Hence

$$\alpha - \alpha' = v(\alpha - \alpha' - dp) + dp\,,$$

where, being $dp \in \overset{k}{\Lambda}_r$ and $\alpha - \alpha' \in \overset{k}{\Lambda}_r$, we have $v(\alpha - \alpha' - dp) \in \overset{k}{\Lambda}_r$. Due to $h \circ v = 0$, we have $[\alpha - \alpha'] = 0$.

Finally, the map is surjective, due to the surjectivity of h. $\boxed{QED}$

Remark 4.1. Let $0 \leq s \leq r$. Then, the sheaf injection χ^r_s induces the sheaf injection

$$\left(\overset{k-n}{\mathcal{C}}{}_s \wedge \overset{n}{\mathcal{H}}{}^h_{s+1}\right) / h(d \ker h) \to \left(\overset{k-n}{\mathcal{C}}{}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}\right) / h(d \ker h)\ .\ \square$$

Theorem 4.2. *Krupka's r–th order variational sequence is isomorphic to the sequence*

$$0 \longrightarrow \overset{0}{\Lambda}_r \xrightarrow{\mathcal{E}_0} \overset{1}{\mathcal{H}}{}^h_r \xrightarrow{\mathcal{E}_1} \dots \xrightarrow{\mathcal{E}_{n-1}} \overset{n}{\mathcal{H}}{}^h_r \xrightarrow{\mathcal{E}_n}$$
$$\left(\overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}\right) / h(d \ker h) \xrightarrow{\mathcal{E}_{n+1}} \dots \xrightarrow{\mathcal{E}_{n+i-1}} \left(\overset{i}{\mathcal{C}}{}^h_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}\right) / h(d \ker h) \xrightarrow{\mathcal{E}_{n+i}} \dots$$

where $\mathcal{E}_0$ coincides with d_h, and $\mathcal{E}_k([h(\alpha)]) = [h(d\alpha)]$. Hence Krupka's variational sequence is isomorphic to the variational sequence associated with the finite order $\mathcal{C}$-spectral sequence, which turns out to be exact.

Theorem 4.3. *(First comparison theorem). We have the identifications*

$$\mathcal{E}_k = \bar{d}\,, \qquad 0 \le k < n\,,$$

so Krupka's finite order variational sequence and the variational sequence associated with the $\mathcal{C}$–spectral sequence coincide up to the degree $k = n$.

PROOF. It comes from the above theorem and the definition of $\bar{d}$. $\boxed{QED}$

Theorem 4.4. *(Second comparison theorem). We have the identifications*

$$\begin{aligned} &\bar{d}(E_0^{p,n-1}) = h(d\ker h)\,,\\ &\tilde{\mathcal{E}}_n(h(\alpha)) = [h(d\alpha)]\,,\\ &\mathcal{E}'_k(h(\alpha)) = [h(d\alpha)]\,, \qquad n < k\,. \end{aligned}$$

where $d\ker h$ stands just for the presheaf $d\ker h$.

PROOF. In fact, we have

$$\bar{d}(E_0^{p,n-1}) = h^{p+1}(d\ker h^p)$$

but $h^p = h$ being $q = n-1$, and $h^{p+1} = h$ on $\overset{p+1+n-1}{\Lambda}{}_r$, hence $\bar{d}(E_0^{p,n-1}) = h(d\ker h)$.

For the two others, we use lemma 3.1, and observe that $\mathcal{E}_n \circ \bar{d} = \mathcal{E}_n \circ \mathcal{E}_{n-1} = 0$, so $\bar{d}(\overset{n-1}{\mathcal{H}}{}^h_r) \subset \ker \mathcal{E}_n$, hence the result follows. $\boxed{QED}$

The above results prove that the two formulations yield the same variational sequence up to the degree n. Indeed, we can improve this result and state the equivalence up to the order $n+1$.

Proposition 4.2. *We have the sequence of presheaf isomorphisms*

$$\Big(\overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}\Big)/h(d\ker h) \simeq \Big(\overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1} + \mathcal{P}\Big) \cap \Big(\overset{1}{\mathcal{C}}_{(2r+1,0)} \wedge \overset{n}{\mathcal{H}}_{2r+1}\Big) \simeq$$
$$\Big(\overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1} + d_h(\overset{1}{\mathcal{C}}_{(2r,r-1)} \wedge \overset{n-1}{\mathcal{H}}{}_{2r})\Big) \cap \Big(\overset{1}{\mathcal{C}}_{(2r+1,0)} \wedge \overset{n}{\mathcal{H}}_{2r+1}\Big) \simeq \Big(\overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}\Big)/\bar{d}(E_0^{p,n-1})\,,$$

where $d\ker h$ stands for the sheaf generated by $d\ker h$ and $\mathcal{P}$ stands for the sheaf generated by $d_h(\overset{1}{\mathcal{C}}_{(2r,r-1)} \wedge \overset{n-1}{\mathcal{H}}{}_{2r})$.

PROOF. The first isomorphism is proved in [28], and it is built essentially by means of the first variation formula, as given in [9]. The first variation formula yields a section of $\mathcal{P}$ for any section of $\overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}$, but, as it is shown in [9], such a section is indeed a section of the presheaf $d_h(\overset{1}{\mathcal{C}}_{(2r,r-1)} \wedge \overset{n-1}{\mathcal{H}}{}_{2r})$ generating $\mathcal{P}$, i.e. , it is of globally of the form $d_h p$, with $p \in \overset{1}{\mathcal{C}}_{(2r,r-1)} \wedge \overset{n-1}{\mathcal{H}}{}_{2r}$. Hence, the second isomorphism holds. The last isomorphism is obtained in the same way of the first one. $\boxed{QED}$

Corollary 4.1. *We have the identification* $\tilde{\mathcal{E}}_n = \mathcal{E}_n$.

By recalling the intrinsic interpretation in terms of the calculus of variations of the variational sequence given in [28], we give the following definition.

Definition 4.1. We say each one of the sheaves of proposition 4.2 to be the *sheaf of Euler–Lagrange morphisms.* □

Remark 4.2. It is very important to note that Krupka's formulation could be modified by using as the contact subsequence the presheaf $\ker h + d \ker h$. This would yield an exact finite order variational sequence (exactness is a local matter), with the unique drawback of the impossibility of computing its cohomology with the de Rham theorem from sheaf theory. *But the sequence obtained in this way would be exactly equal to the variational sequence obtained with the finite order* $\mathcal{C}$*–spectral sequence.* And the cohomology of the last sequence has been compute above! □

5 Infinite order variational sequence

In this section, we analyse the relationships between Krupka's finite order variational bicomplexes of different orders. In particular, we provide a natural inclusion of the variational bicomplex of order $s > 0$ into each variational bicomplex of order $r > s$. Then, we evaluate the direct limit of the system of bicomplexes, obtaining an infinite order variational sequence as the direct limit of the injective system of the finite order variational sequences. As far as we know, this approach is original.

We have the injective system of sheaves

$$\{\overset{k}{\Theta}_s, \pi^{r*}_s\} \ .$$

Lemma 5.1. *[10]. Let* $s \leq r$*. Then, the injective sheaf morphism* π^{r*}_s *induce the injective sheaf morphism*

$$\chi^r_s : \left(\overset{k}{\Lambda}_s/\overset{k}{\Theta}_s\right) \to \left(\overset{k}{\Lambda}_r/\overset{k}{\Theta}_r\right) : [\alpha] \mapsto [\pi^{r*}_s \alpha] \ .$$

Proof. The above morphism χ^r_s is well defined, because

$$[\alpha] = [\beta] \Rightarrow [\pi^{r*}_s \alpha] = [\pi^{r*}_s \beta] \, .$$

The above morphism is also injective, for if $\alpha \in \overset{k}{\Lambda}_s$ and $\beta \in \overset{k}{\Lambda}_s$ are such that

$$[\pi^{r*}_s \alpha] = [\pi^{r*}_s \beta] \ ,$$

then, being $\pi^{r*}_s(\alpha - \beta) \in \pi^{r*}_s \overset{k}{\Lambda}_s$, and $\pi^{r*}_s(\alpha - \beta) \in \overset{k}{\Theta}_r$, we have $\pi^{r*}_s(\alpha - \beta) \in \pi^{r*}_s \overset{k}{\Theta}_s$, hence $[\alpha] = [\beta]$. $\boxed{QED}$

Proposition 5.1. *We have the injective system of sheaves*

$$\{(\overset{k}{\Lambda}_r/\overset{k}{\Theta}_r)\,,\chi^r_s\}\,.$$

Remark 5.1. We have the commutative diagrams

$$\begin{tikzcd} \overset{k}{\Lambda}_r \arrow[r, "d"] & \overset{k+1}{\Lambda}_r \\ \overset{k}{\Lambda}_s \arrow[r, "d"] \arrow[u, "\pi^{r*}_s"] & \overset{k+1}{\Lambda}_s \arrow[u, "\pi^{r*}_s"'] \end{tikzcd} \qquad \begin{tikzcd} \overset{k}{\Theta}_r \arrow[r, "d"] & \overset{k+1}{\Theta}_r \\ \overset{k}{\Theta}_s \arrow[r, "d"] \arrow[u, "\pi^{r*}_s"] & \overset{k+1}{\Theta}_s \arrow[u, "\pi^{r*}_s"'] \end{tikzcd}$$

hence we have the commutative diagram

$$\begin{tikzcd} \overset{k}{\Lambda}_r/\overset{k}{\Theta}_r \arrow[r, "\mathcal{E}_k"] & \overset{k+1}{\Lambda}_r/\overset{k+1}{\Theta}_r \\ \overset{k}{\Lambda}_s/\overset{k}{\Theta}_s \arrow[r, "\mathcal{E}_k"] \arrow[u, "\chi^{r*}_s"] & \overset{k+1}{\Lambda}_s/\overset{k+1}{\Theta}_s \arrow[u, "\chi^{r*}_s"'] \end{tikzcd}$$

We can summarise the above result by stating the existence of a (non exact) three–dimensional commutative diagram, whose bidimensional slices are the variational bicomplexes.

□

We define the presheaves on $\boldsymbol{Y}$

$$\overset{k}{\Theta} := \lim_{\rightarrow} \overset{k}{\Theta}_r\,.$$

Lemma 5.2. *We have*

$$\overset{k}{\Lambda}/\overset{k}{\Theta} = \lim_{\rightarrow} \overset{k}{\Lambda}_r/\overset{k}{\Theta}_r\,.$$

Lemma 5.3. *The sheaf morphisms $\mathcal{E}_k$ induce the presheaf morphisms*

$$\mathcal{E}_k : \left(\overset{k}{\Lambda}/\overset{k}{\Theta}\right) \to \left(\overset{k+1}{\Lambda}/\overset{k+1}{\Theta}\right) : [\alpha] \mapsto [d\alpha]\,,$$

for each $k \geq 0$, where, being $\alpha \in \overset{k}{\Lambda}_r$ for some r, $d\alpha$ coincides with the differential of α on $\overset{k}{\Lambda}_r$.

Theorem 5.1. *The following diagram*

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \overset{1}{\Theta} \arrow[r, "d"] \arrow[d] & \overset{2}{\Theta} \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & \overset{k}{\Theta} \arrow[r, "d"] \arrow[d] & \dots \\
0 \arrow[r] & I\!\!R \arrow[r] \arrow[d] & \overset{0}{\Lambda} \arrow[r, "d"] \arrow[d] & \overset{1}{\Lambda} \arrow[r, "d"] \arrow[d] & \overset{2}{\Lambda} \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & \overset{k}{\Lambda} \arrow[r, "d"] \arrow[d] & \dots \\
0 \arrow[r] & I\!\!R \arrow[r] \arrow[d] & \overset{0}{\Lambda}_r \arrow[r, "\mathcal{E}_0"] \arrow[d] & \overset{1}{\Lambda}/\overset{1}{\Theta} \arrow[r, "\mathcal{E}_1"] \arrow[d] & \overset{2}{\Lambda}/\overset{2}{\Theta} \arrow[r, "\mathcal{E}_2"] \arrow[d] & \dots \arrow[r, "\mathcal{E}_{k-1}"] & \overset{k}{\Lambda}/\overset{k}{\Theta} \arrow[r, "\mathcal{E}_k"] \arrow[d] & \dots \\
 & 0 & 0 & 0 & 0 & & 0 &
\end{tikzcd}
$$

is commutative, and rows and columns are exact presheaf sequences.

PROOF. By the analogous result for finite order variational bicomplexes. $\boxed{QED}$

Note that $\mathcal{E}_0$ coincides with d_h. Moreover, the diagram does not become trivial after a certain value of k, as in the finite order case.

Definition 5.1. The bottom row of the above diagram is said to be the *infinite order variational sequence*. □

6 Infinite order $\mathcal{C}$–spectral sequence

In this section we show that the above infinite order variational sequence can be recovered by means of the $\mathcal{C}$–spectral sequence arising naturally from a fibred manifold (see the Appendix B) [23, 24, 25]. Indeed, we show that the $\mathcal{C}$–spectral sequence induced by the de Rham exact sequence $(\overset{*}{\Lambda}, d)$ (see Lemma 8.1) allows us to recover the infinite order variational sequence.

We recall that the $\mathcal{C}$–spectral sequence is the spectral sequence associated with the cochain complex $(\overset{*}{\Lambda}, d)$ and the graded filtration $(\mathcal{C}^p\overset{*}{\Lambda}, d)_{p\in\mathbb{N}}$, where

$$\mathcal{C}^1\overset{*}{\Lambda} := \{\vartheta \in \overset{*}{\Lambda} \mid \forall\, s \,\text{section of}\, \boldsymbol{Y} \to \boldsymbol{X}\ (js)^*\vartheta = 0\}$$

and $\mathcal{C}^p\overset{*}{\Lambda}$ is the p–th power of the ideal $\mathcal{C}^1\overset{*}{\Lambda}$ in $\overset{*}{\Lambda}$. We set $\mathcal{C}^0\overset{*}{\Lambda} = \{0\}$, and $\mathcal{C}^p\overset{k}{\Lambda} = \overset{k}{\Lambda}$ if $p > k$.

Remark 6.1. We have the injective systems $\{\mathcal{C}^p\overset{k}{\Lambda}_s, \pi^s_r\}$, and

$$\mathcal{C}^p\overset{k}{\Lambda} = \lim_{\rightarrow} \mathcal{C}^p\overset{k}{\Lambda}_s \,.$$

Hence, the computations of the infinite order $\mathcal{C}$–spectral sequence can be performed both by direct evaluation and by direct limit. We will devote little space to proofs in the infinite order case; the interested reader can consult [23, 24, 25]. □

A version of Lemma 3.2 can be given in the infinite order case. Hence, we can describe the presheaves $\mathcal{C}^p\overset{*}{\Lambda}$. The splitting of Theorem 1.3 yields the result in a much simpler way, with respect to the finite order case.

Lemma 6.1. *Let $p \geq 1$. Then, we have*

$$\mathcal{C}^p\overset{p+q}{\Lambda} = \begin{cases} \overset{p}{\mathcal{C}} \wedge \overset{q}{\mathcal{H}} & \textit{if}\, 0 \leq q \leq n-1\,; \\ \overset{p+q}{\Lambda} & \textit{if}\, q \geq n\,. \end{cases}$$

Lemma 6.2. *[25, p.72] We have*

$$E^{p,q}_0 = \overset{p}{\mathcal{C}} \wedge \overset{q}{\mathcal{H}}\,, \qquad \textit{if} \quad 0 \leq q \leq n\,; \qquad E^{p,q}_0 = \{0\} \quad \textit{otherwise;}$$
$$e^{p,q}_0 = d_h : \overset{p}{\mathcal{C}} \wedge \overset{q}{\mathcal{H}} \to \overset{p}{\mathcal{C}} \wedge \overset{q+1}{\mathcal{H}}\,.$$

In other words, the bigraded complex (E_0, e_0) coincides with the bigraded complex $(\overset{*}{\mathcal{C}} \wedge \overset{*}{\mathcal{H}}, d_h)$, where the bigrading is given by the splitting of Theorem 1.3. Anyway, this splitting yields the presheaf morphism d_v too.

Proposition 6.1. *The bigraded complex $(\overset{*}{\mathcal{C}} \wedge \overset{*}{\mathcal{H}}, d_v)$ yields a bigraded complex structure on $\mathcal{C}^*\overset{*}{\Lambda}$ via the equalities of Corollary 6.1. More precisely,*

$$d_v : \overset{p}{\mathcal{C}} \wedge \overset{q}{\mathcal{H}} \to \overset{p+1}{\mathcal{C}} \wedge \overset{q}{\mathcal{H}}\,.$$

We recall that $E_1 = H(E_0)$, where the homology is taken with respect to the presheaf morphism d_h. These two facts yield the following Theorem.

Theorem 6.1. *The above bigraded complexes $(\overset{*}{\mathcal{C}} \wedge \overset{*}{\mathcal{H}}, d_h)$ and $(\overset{*}{\mathcal{C}} \wedge \overset{*}{\mathcal{H}}, d_v)$ yield the* *bicomplex*

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \overset{0}{\Lambda} \arrow[r, "d_v"] \arrow[d, "d_h"'] & \overset{1}{\mathcal{C}} \arrow[r, "d_v"] \arrow[d, "-d_h"'] & \overset{2}{\mathcal{C}} \arrow[r, "d_v"] \arrow[d, "d_h"'] & \dots \\
0 \arrow[r] & \overset{1}{\mathcal{H}} \arrow[r, "d_v"] \arrow[d, "d_h"'] & \overset{1}{\mathcal{C}} \wedge \overset{1}{\mathcal{H}} \arrow[r, "d_v"] \arrow[d, "-d_h"'] & \overset{2}{\mathcal{C}} \wedge \overset{1}{\mathcal{H}} \arrow[r, "d_v"] \arrow[d, "d_h"'] & \dots \\
 & \dots \arrow[d] & \dots \arrow[d] & \dots \arrow[d] & \dots \\
0 \arrow[r] & \dfrac{\overset{n}{\mathcal{H}}}{d_h \overset{n-1}{\mathcal{H}}} \arrow[r, "e_1^{0,n}"] \arrow[d] & \dfrac{\overset{1}{\mathcal{C}} \wedge \overset{n}{\mathcal{H}}}{d_h(\overset{2}{\mathcal{C}} \wedge \overset{n-1}{\mathcal{H}})} \arrow[r, "e_1^{1,n}"] \arrow[d] & \dfrac{\overset{2}{\mathcal{C}} \wedge \overset{n}{\mathcal{H}}}{d_h(\overset{2}{\mathcal{C}} \wedge \overset{n-1}{\mathcal{H}})} \arrow[r, "e_1^{2,n}"] \arrow[d] & \dots \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

which contains the direct limit of the finite order bicomplex arising from the $\mathcal{C}$–spectral sequence on finite order jets.

We have the identifications

$$
\begin{aligned}
E_1^{p,n} &= (\overset{p}{\mathcal{C}} \wedge \overset{n}{\mathcal{H}})/d_h(\overset{p+1}{\mathcal{C}} \wedge \overset{n-1}{\mathcal{H}}) \,, \\
e_1^{p,n} &= \mathcal{E}'_{p+1}
\end{aligned}
$$

Proof. It is easy to see that the direct limit of the finite order bicomplex induced by the $\mathcal{C}$–spectral sequence is constituted by the columns of the above bicomplex together with the bottom row. In particular, one can see that the above presheaf morphisms $e_1^{p,n}$ are the direct limit of the corresponding ones of the finite order case. QED

7 Comparison between infinite order approaches

We evaluate the direct limit of the simplified version of the variational sequences of order r, given in theorem 4.2. Clearly, this limit turns out to be isomorphic to the direct limit of finite order variational sequences.

Remark 7.1. Let $0 \leq s \leq r$. Then, by recalling the injective morphism of remark 4.1, we have the injective system of sheaves

$$
\{\overset{k}{\mathcal{H}}{}^h_s, \pi^r_s\} \quad \text{if} \quad 0 \leq k \leq n \,, \qquad \{\left(\overset{k-n}{\mathcal{C}}{}^h_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}\right)/h(d\ker h) \,, \chi^r_s\} \quad \text{if} \quad n < k \,,
$$

which is isomorphic to $\{\overset{k}{\Lambda}_s/\overset{k}{\Theta}_s, \chi^r_s\}$. □

Lemma 7.1. *The following inclusions hold*

$$h(d\ker h) \subset d_h(\overset{k-n}{\mathcal{C}}{}_r \wedge \overset{n-1}{\mathcal{H}}{}^h_{r+1}) \subset h(\overset{k}{\Theta}_{r+1}) = h(d\ker h)\,.$$

PROOF. By using the decomposition $d = d_h + d_v$. $\boxed{QED}$

Proposition 7.1. *Let $k > 1$. Then, we have the natural isomorphisms*

$$\overset{k}{\Lambda}/\overset{k}{\Theta} \simeq \overset{k}{\mathcal{H}}\,, \qquad if \quad 0 \le k \le n \tag{6}$$

$$\overset{k}{\Lambda}/\overset{k}{\Theta} \simeq \left(\overset{k-n}{\mathcal{C}} \wedge \overset{n}{\mathcal{H}}\right)/d_h(\overset{k-n+1}{\mathcal{C}} \wedge \overset{n-1}{\mathcal{H}}) \qquad if \quad n < k\,. \tag{7}$$

So, the infinite order variational sequence is isomorphic to the following sequence

$$0 \longrightarrow \overset{0}{\Lambda} \xrightarrow{\mathcal{E}_0} \overset{1}{\mathcal{H}} \xrightarrow{\mathcal{E}_1} \dots \xrightarrow{\mathcal{E}_{n-1}} \overset{n}{\mathcal{H}} \xrightarrow{\mathcal{E}_n}$$

$$(\overset{1}{\mathcal{C}} \wedge \overset{n}{\mathcal{H}})/d_h(\overset{2}{\mathcal{C}} \wedge \overset{n-1}{\mathcal{H}}) \xrightarrow{\mathcal{E}_{n+1}} \dots \xrightarrow{\mathcal{E}_{n+i-1}} (\overset{i}{\mathcal{C}} \wedge \overset{n}{\mathcal{H}})/d_h(\overset{i+1}{\mathcal{C}} \wedge \overset{n-1}{\mathcal{H}}) \xrightarrow{\mathcal{E}_{n+i}} \dots$$

where $\mathcal{E}_k$ coincides with d_h if $0 \le k \le n$, and $\mathcal{E}_k([\alpha]) = [d_v(\alpha)]$ if $k > n$.

PROOF. In fact, the isomorphisms (7) come from the above lemma. We have to prove that $\mathcal{E}_k([\alpha]) = [d_v(\alpha)]$. But we have $\alpha = h(\beta)$, and

$$\mathcal{E}_k([\alpha]) = \mathcal{E}_k([h(\beta)]) = [h(d\beta)]\,,$$

with

$$h(d\beta) = h((d_h + d_v)(h(\beta) + v(\beta)) = d_v(h(\beta)) + d_h(v(\beta))\,,$$

hence the result. $\boxed{QED}$

Thus, we have provided the infinite order analogue (indeed, the direct limit) of the sequence of Theorem 4.2. As for the comparison between the above sequence and the infinite order variational sequences associated with the $\mathcal{C}$–spectral sequence, we note that the results of theorems4.3, 4.4 and proposition 4.2, and even remark 4.2 hold in the direct limit.

Theorem 7.1. *The infinite order variational sequence provided by the direct limit of Krupka's variational sequence and the infinite order variational sequences associated with the $\mathcal{C}$–spectral sequence are isomorphic up to the degree $n+1$.*

In particular, the space of infinite order Euler–Lagrange morphism turn out to be $\overset{1}{\mathcal{C}}_{(*,0)} \wedge \overset{n}{\mathcal{H}}$, where $\overset{1}{\mathcal{C}}_{(*,0)} := \varinjlim \overset{1}{\mathcal{C}}_{(r,0)}$.

PROOF. The first part comes from the above quoted results, and the last assertion comes from the following inclusions

$$\left(\overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1} + \mathcal{P}\right) \cap \left(\overset{1}{\mathcal{C}}_{(2r+1,0)} \wedge \overset{n}{\mathcal{H}}_{2r+1}\right) \subset$$

$$\subset \overset{1}{\mathcal{C}}_{(2r+1,0)} \wedge \overset{n}{\mathcal{H}}_{2r+1} \subset$$

$$\subset \left(\overset{1}{\mathcal{C}}_{2r+1} \wedge \overset{n}{\mathcal{H}}{}^h_{2r+2} + \mathcal{P}\right) \cap \left(\overset{1}{\mathcal{C}}_{(4r+3,0)} \wedge \overset{n}{\mathcal{H}}_{4r+3}\right)\,. \quad \boxed{QED}$$

8 Conclusions

We have shown what are the relationships between two of the most important geometric formulations of Lagrangian formalism. Moreover, we provided two new formulations, each of which is inspired by one of the above two.

We stress that each formulation can be carried on independently, giving rise to two exact sequences with the same cohomologies, and any of the two yields the same information for the Lagrangian formalism up to the degree $n+1$.

As for the degree $n+2$, we recall [10] that this yields information on the local variationality of Euler–Lagrange operators, and there exists an intrinsic formuations of the conditions of local variationality (Helmholz morphism, see [28]). At the present moment we are studying the equivalence of the sequence at the degree $n+2$, and we found an equivalence up to the order $r=2$. We stress that there is no interpretation in terms of the geometric objects of the calculus of variations for sections having degree $k > n+2$.

Appendix A: direct sums and exterior products

Let V be a vector space such that $\dim V = n$. We recall that the *box product* (see, for example, [7]) of r linear morphisms $a_1, \dots a_r : V \to V$ is defined to be the linear map

$$\begin{aligned} a_1 \Box \dots \Box a_r : \overset{r}{\wedge} V \to \overset{r}{\wedge} V : \\ v_1 \wedge \dots \wedge v_r \mapsto \sum_{\sigma \in S_r} |\sigma| a_1(v_{\sigma(1)}) \wedge \dots \wedge a_r(v_{\sigma(r)}) \,. \end{aligned}$$

where S_r is the set of all permutation of order r. The box product fulfills

$$a_1 \Box \dots \Box a_r = a_{\sigma(1)} \Box \dots \Box a_{\sigma(r)} \qquad \forall\, \sigma \in S_r \,,$$
$$a \Box \dots \Box a = r! \overset{r}{\wedge} a \,;$$

so, $\Box$ yields a map $\overset{k}{\odot}(\mathrm{End}(V)) \to \mathrm{End}(\overset{k}{\wedge} V)$.

We have a remarkable feature of the box product. Suppose that $V = W_1 \oplus W_2$, with $p_1 : V \to W_1$ and $p_2 : V \to W_2$ the related projections. Then, we have the splitting

$$\overset{m}{\wedge} V = \bigoplus_{k+h=m} \overset{k}{\wedge} W_1 \wedge \overset{h}{\wedge} W_2 \,, \tag{8}$$

where $\overset{k}{\wedge} W_1 \wedge \overset{h}{\wedge} W_2$ is the subspace of $\overset{m}{\wedge} V$ generated by the wedge products of elements of $\overset{k}{\wedge} W_1$ and $\overset{h}{\wedge} W_2$. The projections $p_{k,h}$ related to the above splitting turn out to be the maps

$$p_{k,h} = \frac{1}{k!\, h!} \Box^k p_1 \Box \Box^h p_2 : \overset{m}{\wedge} V \to \overset{k}{\wedge} W_1 \wedge \overset{h}{\wedge} W_2 \,.$$

Remark 8.1. Let $V' \subset V$ be a vector subspace, and set $W_1' := p_1(V')$, $W_2' := p_2(V')$. Then we have

$$V' \subset W_1' \oplus W_2',$$

but the inclusion, in general, is not an equality.

Appendix B: spectral sequences

In this section, we give the basic material on spectral sequences. In the first subsection we recall the definition of spectral sequence, togheter with some preliminar concepts. In the second subsection we give the notions of exact couple and derived couple. The third subsection is devoted to the definition of spectral sequence associated with a filtration of a given complex. The interest reader can consult [3, 12, 14, 18] for more details and applications.

Spectral sequences

In this subsection we give some preliminar definitions. Note that we will introduce graded groups and maps with degrees in $\mathbb{N}$ rather than $\mathbb{Z}$. This is due to the fact that in our applications we will not need a grading in $\mathbb{Z}$.

Definition 8.1. A *differential group* is defined to be a pair (Λ, d), where Λ is an Abelian group and $d : \Lambda \to \Lambda$ is a group morphism such that $d^2 = 0$, or, equivalently, $\operatorname{im} d \subset \ker d$.

The morphism d is said to be the *differential* of Λ.

The *homology* of the differential group is defined to be the abelian group

$$H(\Lambda) := \ker d \big/ \operatorname{im} d\,. \quad \square$$

Definition 8.2. A *graded differential group (of degree g)* is defined to be a pair $(\overset{*}{\Lambda}, d)$, where

$$\overset{*}{\Lambda} := \oplus_{k\in\mathbb{N}} \overset{k}{\Lambda}$$

is a graded Abelian group and $d : \overset{*}{\Lambda} \to \overset{*}{\Lambda}$ is a graded morphism of degree g, i.e.

$$d(\overset{k}{\Lambda}) \subset \overset{k+g}{\Lambda},$$

such that $d^2 = 0$, or, equivalently, $\operatorname{im} d \subset \ker d$. $\square$

We recall that a *cochain complex* is a sequence of morphisms of abelian groups of the form

$$0 \longrightarrow \overset{0}{\Lambda} \xrightarrow{d_0} \overset{1}{\Lambda} \xrightarrow{d_1} \overset{2}{\Lambda} \xrightarrow{d_2} \dots$$

such that $d_{k+1} \circ d_k = 0$. This last condition is equivalent to $\operatorname{im} d_k \subset \ker d_{k+1}$. A cochain complex is said to be an *exact sequence* if $\operatorname{im} d_k = \ker d_{k+1}$. To each cochain complex we can define the *cohomology group*

$$H^*(\overset{*}{\Lambda}) = \oplus_{k \in \mathbb{N}} H^k(\overset{*}{\Lambda}) \,,$$

where

$$H^k(\overset{*}{\Lambda}) := (\ker d_k) \big/ (\operatorname{im} d_{k-1}) \,.$$

The cohomology groups vanish if and only if the cochain complex is an exact sequence.

Lemma 8.1. *There is a bijection between graded differential groups* $(\overset{*}{\Lambda}, d)$ *of degree* $+1$ *and cochain complexes*

$$0 \longrightarrow \overset{0}{\Lambda} \xrightarrow{\ d\ } \overset{1}{\Lambda} \xrightarrow{\ d\ } \overset{2}{\Lambda} \xrightarrow{\ d\ } \dots$$

Moreover, the homology of $(\overset{*}{\Lambda}, d)$ *coincides with the cohomology of the corresponding cochain complex.*

So, we identify any graded differential group $(\overset{*}{\Lambda}, d)$ of degree $+1$ with the cochain complex associated with $(\overset{*}{\Lambda}, d)$ via the above Lemma.

Definition 8.3. We define a *spectral sequence* to be a sequence of differential groups

$$(E_n, e_n)_{n \in \mathbb{N}}$$

such that

$$E_{n+1} = H(E_n) \,.$$

We say that the spectral sequence converges to E_r if $E_r = E_k$ for any $k > r$. □

Exact couples

Definition 8.4. An *exact couple* is defined to be a pair (Q, S) of abelian groups togheter with an exact sequence of morphisms

$$\begin{tikzcd} S \arrow[rr, "i"] & & S \arrow[dl, "\pi"] \\ & Q \arrow[ul, "\delta"] & \end{tikzcd}$$

□

Remark 8.2. If (Q, S) is an exact couple as above, then the pair (Q, e), where $e := \pi \circ \delta$, is a differential group. In fact, $(\pi \circ \delta)^2 = 0$ due to the exactness of the above diagram. $\square$

Proposition 8.1. *Let (Q, S) be an exact couple, as in the above definition. Then, the pair (E_1, S_1), where*

$$E_1 := H(Q)\,, \qquad S_1 := i(S)\,,$$

together with the diagram

$$\begin{tikzcd} S_1 \arrow[rr, "i_1"] & & S_1 \arrow[dl, "\pi_1"] \\ & Q_1 \arrow[ul, "\delta_1"] & \end{tikzcd}$$

where

$$\begin{aligned} i_1 &: i(S) \to i(S) : i(s) \mapsto i(i(s))\,, \\ \pi_1 &: i(S) \to H(Q) : i(s) \mapsto [\pi(s)]\,, \\ \delta_1 &: H(Q) \to i(S) : [q] \mapsto \delta(q)\,, \end{aligned}$$

is an exact couple.

Proof. One has to check that the above maps are well defined, and that the above diagram is commutative and exact. This is straightforward. $\boxed{QED}$

The above exact couple is said to be the *derived couple*. The pair (E_1, e_1), where $e_1 := \pi_1 \circ \delta_1$, turn out to be a differential group.

We can consider iterated derived couples; namely, we set by induction

$$\begin{aligned} &(E_0, S_0) := (E, S)\,; \\ &(E_{n+1}, S_{n+1}) := ((E_n)_1, (S_n)_1) \qquad \forall\, n > 0\,; \end{aligned}$$

analogously, we define $i_n, \pi_n, \delta_n, e_n$. So, we have the sequence of differential groups $(E_n, e_n)_{n\in\mathbb{N}}$, and the following obvious result.

Proposition 8.2. *Any exact couple (E, Q) yields a spectral sequence $(E_n, e_n)_{n\in\mathbb{N}}$.*

Remark 8.3. We remark that, if Q and S are graded abelian groups, i, π are graded morphisms of degree 0 and δ is a graded morphism of degree $+1$, then $(E_n, e_n)_{n\in\mathbb{N}}$ is a spectral sequence which is made by graded differential groups. $\square$

Filtered differential groups

Let (Λ, d) be a differential group. A *differential subgroup* is defined to be a differential group $(\mathcal{C}, d')$ such that $\mathcal{C} \subset \Lambda$ is an abelian subgroup and $d' = d|_{\mathcal{C}}$. We will denote d' by d, by an abuse of notation.

Definition 8.5. We define a *filtration* of a differential group (Λ, d) to be a sequence of differential subgroups $(\mathcal{C}^p, d)_{p\in\mathbb{N}}$ of (Λ, d), where $\mathcal{C}^0 := \Lambda$, which is decreasing with respect to the inclusion, namely

$$\Lambda \equiv \mathcal{C}^0 \supset \mathcal{C}^1 \supset \mathcal{C}^2 \supset \dots$$

If there exists $l \in \mathbb{N}$ such that $\mathcal{C}^l \neq \{0\}$ but $\mathcal{C}^k = \{0\}$ for $k > l$, then we say that the filtration has *finite length* l.

If $(\mathcal{C}^p, d)_{p\in\mathbb{N}}$ is a filtration of (Λ, d), then we say (Λ, d) to be a *filtered differential group*. □

Let $(\mathcal{C}^p, d)_{p\in\mathbb{N}}$ be a filtration of (Λ, d). We define the abelian groups

$$Q^p := \mathcal{C}^p / \mathcal{C}^{p+1}, \qquad Q := \oplus_{p\in\mathbb{N}} Q^p .$$

Lemma 8.2. *For each $p \geq 0$, the morphism d passes to the quotient $\mathcal{C}^p/\mathcal{C}^{p+1}$. The induced graded morphism of degree 0 $\bar{d} : Q \to Q$ fulfills $\bar{d}^2 = 0$. Hence, we have the graded differential group (of degree 0) $(Q, \bar{d})$.*

The pair $(Q, \bar{d})$ is said to be the *graded differential group associated with the filtration*. Moreover, we define the graded differential group (of degree 0)

$$S := \oplus_{p\in\mathbb{N}} \mathcal{C}^p .$$

Lemma 8.3. *We have the graded exact sequence of graded differential groups*

$$0 \longrightarrow S \xrightarrow{\ i\ } S \xrightarrow{\ \pi\ } Q \longrightarrow 0$$

where $i|_{\mathcal{C}^{p+1}} : \mathcal{C}^{p+1} \to \mathcal{C}^p$ is the inclusion map, of degree -1, and

$$\pi|_{\mathcal{C}^p} : \mathcal{C}^p \to \mathcal{C}^p / \mathcal{C}^{p+1}$$

is the natural projection, of degree 0. The maps i, π commute with the differentials in the domains and codomains.

Passing to cohomologies, we obtain the exact sequence

$$\dots \longrightarrow H^k(S) \xrightarrow{\ i\ } H^k(S) \xrightarrow{\ \pi\ } H^k(Q) \xrightarrow{\ \delta\ } H^{k+1}(S) \longrightarrow \dots$$

which yields the exact couple

$$\begin{tikzcd} H^*(S) \arrow[rr, "i"] & & H^*(S) \arrow[dl, "\pi"] \\ & H^*(Q) \arrow[ul, "\delta"] & \end{tikzcd}$$

where δ is the Bockstein operator, of degree $+1$, and the graded differential group $(H^*(Q), e)$, where $e := \pi \circ \delta$ has degree $+1$, and $e^2 = 0$. We stress that $i : H^*(S) \to H^*(S)$ is no longer the inclusion map.

Remark 8.4. We recall the definition of the Bockstein operator in this context.

Let $[\alpha] \in H^k(Q)$. Then, being π surjective, we choose $\beta \in S^k \equiv \mathcal{C}^k$ such that $\pi(\beta) = \alpha$. We see that $\pi(d\beta) = \bar{d}\pi(\beta) = 0$, hence due to the exactness, there exists a unique $\gamma \in S^{k+1} = \mathcal{C}^{k+1}$ such that $i(\gamma) = d\beta$ (actually, $\gamma = d\beta$, because i is the inclusion map). Finally, $d\beta$ is closed in S^{k+1}, due to $d^2\beta = 0$, but it is not exact in S^{k+1}, hence it determines a class $[d\beta] \in H^{k+1}(S)$. We can easily prove that

$$\delta : H^k(Q) \to H^{k+1}(S) : [\alpha] \mapsto [d\beta]$$

is well defined. □

Theorem 8.1. *Let (Λ, d) be a differential group. Then, each filtration $(\mathcal{C}^p, d)_{p\in\mathbb{N}}$ of (Λ, d) induces a graded spectral sequence $(E_n^*, e_n^*)_{n\in\mathbb{N}}$ (see Remark 8.3) as follows*

$$\begin{aligned} &E_0 := Q\,, \qquad e_0 := \bar{d}\,; \\ &(E_1^*, S_1^*) := (H^*(Q), H^*(S))\,, \qquad e_1 := e \equiv \pi \circ \delta\,; \\ &(E_n^*, S_n^*) := (H^*(E_n), i^{n-1}(H^*(S)))\,, \qquad e_n := \pi_n \circ \delta_n \end{aligned}$$

Note that (E_0, S) is not an exact couple, but (E_0, e_0) is a graded differential group (of degree 1).

Definition 8.6. Let (Λ, d) be a differential group with a given filtration. We say $(E_n^*, e_n^*)_{n\in\mathbb{N}}$ to be the (graded) *spectral sequence associated with the filtered differential group* (Λ, d). □

Remark 8.5. We have an important particular case of filtered differential group. Namely, suppose that $(\overset{*}{\Lambda}, d)$ is a graded differential group (of degree $+1$), and $(\overset{*}{\mathcal{C}}{}^p, d)_{p\in\mathbb{N}}$ is a graded filtration, i.e. a filtration by graded differential subgroups whose grading is compatible with the grading of $(\overset{*}{\Lambda}, d)$.

The spectral sequence associated with $(\overset{*}{\Lambda}, d)$ is a sequence of bigraded complexes $(E_n^{*,*}, e_n^{*,*})$. More precisely, we have the bigraded differential groups

$$S^{*,*} := \oplus_{p,q\in\mathbb{N}} \overset{p+q}{\mathcal{C}}{}^p\,, \qquad Q^{*,*} := \oplus_{p,q\in\mathbb{N}} \overset{q}{\mathcal{C}}{}^p \big/ \overset{q-1}{\mathcal{C}}{}^{p+1}\,,$$
$$E_n^{*,*} := \bigoplus_{p,q\in\mathbb{N}} E_n^{p,q}\,,$$

where p is the filtration degree and $p+q$ is the degree induced by $\overset{*}{\Lambda}$; q is said to be the *complementary degree*. The morphisms i, π, δ turn out to be bigraded morphisms with bidegrees $(-1,+1)$, $(0,0)$, $(+1,0)$ respectively. Moreover, it can be proved that the maps i_n, π_n, δ_n have bidegrees $(1,-1)$, $(n-1,-n+1)$, $(+1,0)$, respectively, hence

$$e_n^{p,q} : E_n^{p,q} \to E_n^{p+n,q-n+1}\,. \ \square$$

As for the graded case we have a very important result.

Theorem 8.2. *Let* $(\overset{*}{\Lambda}, d)$ *be a graded differential group (of degree* $+1$*) with a graded filtration* $(\overset{*}{\mathcal{C}}{}^p, d)_{p\in\mathbb{N}}$*. Suppose that to any degree* $n \in \mathbb{N}$ *the filtration* $(\overset{n}{\mathcal{C}}{}^p, d)_{p\in\mathbb{N}}$ *has finite length. Then, the spectral sequence induced by the filtration converges to* $H^*(\overset{*}{\Lambda})$.

PROOF. It can be easily deduced from the definitions [3, p. 160]. $\boxed{QED}$

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