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Finite order variational bicomplexes

Raffaele Vitolo[1]

Dept. of Mathematics "E. De Giorgi", Università di Lecce,

via per Arnesano, 73100 Italy

email: Raffaele.Vitolo@unile.it

1 Introduction

The theory of variational bicomplexes was established at the end of the seventies by several authors [2, 17, 23, 26, 29, 30, 31, 32]. The idea is that the operations which take a Lagrangian into its Euler–Lagrange morphism [9, 10, 12, 24] and an Euler–Lagrange morphism into its Helmholtz' conditions of local variationality [2, 1, 3, 7, 11, 18, 13, 27] are morphisms of a (long) exact sheaf sequence. This viewpoint allows to overcome several problems of Lagrangian formulations in mechanics and field theories [21, 28]. To avoid technical difficulties variational bicomplexes were formulated over the space of infinite jets of a fibred manifold. But in this formalism the information relatively to the order of the jet where objects are defined is lost.

We refer to the recent formulation of variational bicomplexes on finite order jet spaces [13]. Here, a finite order variational sequence is obtained by quotienting the de Rham sequence on a finite order jet space with an intrinsically defined subsequence, whose choice is inspired by the calculus of variations. It is important to find an isomorphism of the quotient sequence with a sequence of sheaves of 'concrete'sections of some vector bundle. This task has already been faced locally [22, 25] and intrinsically [33] in the case of one independent variable.

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In this paper, we give an *intrinsic* isomorphism of the variational sequence (in the general case of n independent variables) with a sequence which is made by sheaves of forms on a jet space of minimal order. This yields new natural solutions to problems like the minimal order Lagrangian corresponding to a locally variational Euler–Lagrange morphism and the search of variationally trivial Lagrangians. Moreover, we give a new intrinsic formulation of Helmholtz' local variationality conditions, proving the existence of a new intrinsic geometric object which, for an Euler–Lagrange morphism, plays a role analogous to that of the momentum of a Lagrangian.

We observe that the finite order variational bicomplexes provide a unique comprehensive framework (or better, a *language*) for dealing with Lagrangian theories. The mathematical (or 'metaphysical', see [28]) problems which arise during the formulation of any Lagrangian theory can be understood and solved by means of the finite order variational bicomplexes. Moreover, the algebraic methods used throughout the paper allow a synthetic and clear understanding of concepts whose meaning could hardly be reached by means of coordinate expressions alone. See [21] for a first application of the first order variational bicomplex to a relativistic theory of mechanics.

Throughout the paper, we will use as fundamental tools the structure form on jet spaces [19], and the horizontal and vertical differential [24]. Moreover, we make use of intrinsic techniques that are developed by means of the language of [6], and which were first introduced in [12].

Manifolds and maps between manifolds are C^∞. All morphisms of fibred manifolds (and hence bundles) will be morphisms over the identity of the base manifold, unless otherwise specified. As for sheaves, we will use the definitions and the main results given in [36]. In particular, we will be concerned only with sheaves of $\mathbb{R}$–vector spaces. Thus, by 'sheaf morphism' we will mean morphism of sheaves of $\mathbb{R}$–vector spaces.

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Diagrams have been drawn by P. Taylor's `diagrams` macro package.

2 Variational bicomplexes on finite order jet spaces

In this section, we first recall some basic facts on jet spaces [19, 17, 24]. Then, we recall Krupka's formulation of the finite order variational bicomplex [13].

Jet spaces

The main purpose of this section is to introduce jet spaces of a fibred manifold and study some properties of the sheaf of forms on the r–th order jet space. Moreover, we recall the horizontal and vertical differential [24].

Our framework is a fibred manifold

$$\pi : \boldsymbol{Y} \to \boldsymbol{X} \,,$$

with $\dim \boldsymbol{X} = n$ and $\dim \boldsymbol{Y} = n + m$.

For $0 \leq r$, we are concerned with the r–jet space $J_r\boldsymbol{Y}$; in particular, we set $J_0\boldsymbol{Y} \equiv \boldsymbol{Y}$. We recall the natural fibrings $\pi^r_s : J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$, $\pi^r : J_r\boldsymbol{Y} \to \boldsymbol{X}$, and, among these, the affine fibring π^r_{r-1}.

Charts on $\boldsymbol{Y}$ adapted to π are denoted by (x^λ, y^i). Greek indices $\lambda, \mu, \ldots$ run from 1 to n and label base coordinates, Latin indices $i, j, \ldots$ run from 1 to m and label fibre coordinates, unles otherwise specified. We denote by $(\partial_\lambda, \partial_i)$ and (d^λ, d^i), respectively, the local bases of vector fields and 1–forms on $\boldsymbol{Y}$ induced by an adapted chart.

We denote multi–indices of dimension n by underlined latin letters such as $\underline{p} = (p_1, \ldots, p_n)$, with $0 \leq p_1, \ldots, p_n$; we identify the index λ with the multi–index $(p_\mu) = (\delta^\lambda_\mu)$. We also set $|\underline{p}| := p_1 + \cdots + p_n$ and $\underline{p}! := p_1! \ldots p_n!$.

The charts induced on $J_r\boldsymbol{Y}$ are denoted by $(x^0, y^i_{\underline{p}})$, with $0 \leq |\underline{p}| \leq r$; in particular, we set $y^i_{\underline{0}} \equiv y^i$. A section $s : \boldsymbol{X} \to \boldsymbol{Y}$ can be prolonged to a section $j_rs : \boldsymbol{X} \to J_r\boldsymbol{Y}$, with coordinate expression $(j_rs)^i_{\underline{p}} = \partial^i_{\underline{p}} s^i$. The local vector fields and forms of $J_r\boldsymbol{Y}$ induced by the fibre coordinates are denoted by $(\partial^{\underline{p}}_i)$ and $(d^i_{\underline{p}})$, respectively. A vertical vector field on $\boldsymbol{Y}$ can be prolonged to a vertical vector field on $J_r\boldsymbol{Y}$ (see [19, 24]).

A fundamental role is played in the theory of variational bicomplexes by the *contact maps* on jet spaces (see [19]). Namely, for $1 \leq r$, we consider the natural complementary fibred morphisms over $J_r\boldsymbol{Y} \to J_{r-1}\boldsymbol{Y}$

$$Д_r : J_r\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X} \to TJ_{r-1}\boldsymbol{Y} \,, \qquad \vartheta_r : J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} TJ_{r-1}\boldsymbol{Y} \to VJ_{r-1}\boldsymbol{Y} \,,$$

whose coordinate expression are

$$Д_r = d^\lambda \otimes Д_{r\lambda} = d^\lambda \otimes (\partial_\lambda + y^j_{\underline{p}+\lambda} \partial^{\underline{p}}_j) \,, \qquad \vartheta_r = \vartheta^j_{\underline{p}} \otimes \partial^{\underline{p}}_j = (d^j_{\underline{p}} - y^j_{\underline{p}+\lambda} d^\lambda) \otimes \partial^{\underline{p}}_j \,,$$

for $0 \leq |\underline{p}| \leq r - 1$. We have the splitting [19]

$$(1) \qquad J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*J_{r-1}\boldsymbol{Y} = \left(J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*\boldsymbol{X} \right) \oplus \operatorname{im} \vartheta^*_r \,.$$

We are concerned with some distinguished sheaves of forms on jet spaces. Note that we will consider sheaves on $J_r\boldsymbol{Y}$ with respect to the topology generated by open sets of the kind $(\pi^r_0)^{-1}(\boldsymbol{U})$, with $\boldsymbol{U} \subset \boldsymbol{Y}$ open in $\boldsymbol{Y}$. This is due to the topological triviality of the fibre of $J_r\boldsymbol{Y} \to J_{r-1}\boldsymbol{Y}$.

Let $0 \leq k$.

i. For $0 \leq r$, we consider the standard sheaf $\overset{k}{\Lambda}_r$ of k–forms on $J_r\boldsymbol{Y}$.

ii. For $0 \leq s \leq r$, we consider the sheaves $\overset{k}{\mathcal{H}}_{(r,s)}$ and $\overset{k}{\mathcal{H}}_r$ of *horizontal forms*, i.e. of local fibred morphisms over $J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$ and $J_r\boldsymbol{Y} \to \boldsymbol{X}$ of the type $\alpha : J_r\boldsymbol{Y} \to \overset{k}{\wedge} T^*J_s\boldsymbol{Y}$ and $\beta : J_r\boldsymbol{Y} \to \overset{k}{\wedge} T^*\boldsymbol{X}$, respectively. If $0 \leq q \leq r$, then pull–back by π^r_q provides several inclusions; for example, we have $\overset{k}{\mathcal{H}}_q \subset \overset{k}{\mathcal{H}}_r$ and $\overset{k}{\Lambda}_q \subset \overset{k}{\mathcal{H}}_{(r,q)}$. We have the distinguished subsheaf $\overset{k}{\mathcal{H}}{}^P_r \subset \overset{k}{\mathcal{H}}_r$ of local fibred morphisms $\alpha \in \overset{k}{\mathcal{H}}_r$ such that α is a polynomial fibred morphism over $J_{r-1}\boldsymbol{Y} \to \boldsymbol{X}$ of degree k.

iii. For $0 \leq s < r$, we consider the subsheaf $\overset{k}{\mathcal{C}}_{(r,s)} \subset \overset{k}{\mathcal{H}}_{(r,s)}$ of *contact forms*, i.e. of sections $\alpha \in \overset{k}{\mathcal{H}}_{(r,s)}$ with values into $\overset{k}{\wedge}\mathrm{im}\,\vartheta^*_{s+1}$. Due to the injectivity of ϑ^*_{s+1}, any section $\alpha \in \overset{k}{\mathcal{C}}_{(r,s)}$ factorises as $\alpha = \overset{k}{\wedge}\vartheta^*_{s+1} \circ \tilde{\alpha}$, where $\tilde{\alpha}$ is a section of $J_r\boldsymbol{Y} \underset{J_s\boldsymbol{Y}}{\times} \overset{k}{\wedge} V^*J_s\boldsymbol{Y} \to J_s\boldsymbol{Y}$. We have the distinguished subsheaf $\overset{k}{\mathcal{C}}_r \subset \overset{k}{\mathcal{C}}_{(r+1,r)}$ of local fibred morphisms $\alpha \in \overset{k}{\mathcal{C}}_{(r+1,r)}$ such that $\tilde{\alpha}$ projects down on $J_r\boldsymbol{Y}$.

The fibred splitting (1) yields the sheaf splitting

$$\overset{k}{\mathcal{H}}_{(r+1,r)} = \bigoplus_{l=0}^{k} \overset{k-l}{\mathcal{C}}_{(r+1,r)} \wedge \overset{l}{\mathcal{H}}_{r+1}\,, \tag{2}$$

We set H to be the projection of the above splitting on the nontrivial summand with the greatest value of l. We set also $V := Id - H$.

Now, we want to find the image of $\overset{k}{\Lambda}_r \subset \overset{k}{\mathcal{H}}_{(r+1,r)}$ under the projections of the above splitting. We denote the restrictions of H, V to $\overset{k}{\Lambda}_r$ by h, v.

Proposition 2.1. *Let $0 < k \leq n$, and set*

$$\overset{k}{\mathcal{H}}{}^h_{r+1} := h(\overset{k}{\Lambda}_r)\,.$$

Then, we have the inclusion $\overset{k}{\mathcal{H}}{}^h_{r+1} \subset \overset{k}{\mathcal{H}}{}^P_{r+1}$. More precisely, if $\alpha \in \overset{k}{\mathcal{H}}{}^P_{r+1}$, then

$\alpha \in \overset{k}{\mathcal{H}}{}^h_{r+1}$ if and only if there exists $\beta \in \overset{k}{\Lambda}_r$ such that $(j_r s)^*\beta = (j_{r+1}s)^*\alpha$ for each section $s : \boldsymbol{X} \to \boldsymbol{Y}$.

Proof. It comes from the identities $(j_r s)^*\beta = (j_{r+1}s)^* h(\beta)$, $(j_{r+1}s)^* v(\beta) = 0$. QED

Note that, if $\dim \boldsymbol{X} = 1$, then we have $\overset{1}{\mathcal{H}}{}^h_{r+1} = \overset{1}{\mathcal{H}}{}^P_{r+1}$.

Proposition 2.2. *The splitting* (2) *yields the inclusion*

$$\overset{k}{\Lambda}_r \subset \bigoplus_{l=0}^{k} \overset{k-l}{\mathcal{C}}_r \wedge \overset{l}{\mathcal{H}}{}^h_{r+1}\,,$$

and the splitting projections (and hence h) restrict to surjective maps.

Proof. The above inclusion can be easily checked in coordinates. Then, a partition of unity argument shows that the splitting projections are surjective. QED

We remark that, in general, the above inclusion is a proper inclusion: in general, a sum of elements of the direct summands is not an element of $\overset{k}{\Lambda}_r$.

We have two remarkable derivations of degree one (see [24, 6]). Namely, we define the *horizontal* and *vertical differential* to be the sheaf morphisms

$$d_h := [i_{\text{д}}, d] : \overset{k}{\Lambda}_r \to \overset{k}{\Lambda}_{r+1}\,, \qquad d_v := [i_\vartheta, d] : \overset{k}{\Lambda}_r \to \overset{k}{\Lambda}_{r+1}\,,$$

It can be proved (see [24]) that d_h and d_v fulfill the property $d_h + d_v = (\pi^{r+1}_r)^* \circ d$.
The action of d_h and d_v on functions $f : J_r\boldsymbol{Y} \to I\!R$ and one–forms on $J_r\boldsymbol{Y}$ uniquely characterises d_h and d_v. We have the coordinate expressions

$$d_h f = (\partial_\lambda f + y^i_{\underline{p}+\lambda}\partial^{\underline{p}}_i f) d^\lambda\,, \qquad d_h d^\lambda = 0\,, \qquad d_h \vartheta^i_{\underline{p}} = -\vartheta^i_{\underline{p}+\lambda} \wedge d^\lambda\,,$$
$$d_v f = \partial^{\underline{p}}_i f \vartheta^i_{\underline{p}}\,, \qquad d_v d^\lambda = 0\,, \qquad d_v \vartheta^i_{\underline{p}} = 0\,.$$

hence the inclusions

$$d_h\left(\overset{k}{\mathcal{C}}_{(r,r-1)} \wedge \overset{h}{\mathcal{H}}_r\right) \subset \overset{k}{\mathcal{C}}_{(r+1,r)} \wedge \overset{h+1}{\mathcal{H}}_{r+1}\,, \quad d_v\left(\overset{k}{\mathcal{C}}_{(r,r-1)} \wedge \overset{h}{\mathcal{H}}_r\right) \subset \overset{k+1}{\mathcal{C}}_{(r+1,r)} \wedge \overset{h}{\mathcal{H}}_{r+1}\,.$$

Finite order variational bicomplex

Here, we recall the theory of variational bicomplexes on finite order jet spaces, as was developed by Krupka in [13]. We have a natural exact subsequence of the de Rham sequence on $J_r\boldsymbol{Y}$. This subsequence is not the unique exact and natural one that we might consider; our choice is inspired by the calculus of variations. Then we define the (r–th order) variational sequence to be the quotient of the de Rham sequence on $J_r\boldsymbol{Y}$ by means of the above exact subsequence.

Let us denote by $d\ker h$ the sheaf generated by the presheaf $d\ker h$, by an abuse of notation. We set $\overset{k}{\Theta}_r := \ker h + d\ker h$; we have the following natural subsequence of the de Rham sequence on $J_r\boldsymbol{Y}$

$$0 \longrightarrow \overset{1}{\Theta}_r \xrightarrow{\;d\;} \overset{2}{\Theta}_r \xrightarrow{\;d\;} \dots \xrightarrow{\;d\;} \overset{I}{\Theta}_r \xrightarrow{\;d\;} 0$$

In general, I depends on the dimension of the fibers of $J_r\boldsymbol{Y} \to \boldsymbol{X}$ [13]. It is proved in [13] that the contact subsequence is exact, and the sheaves $\overset{k}{\Theta}_r$ are soft.

If $0 \leq k \leq n$, then $d\ker h \subset \ker h$, and

$$\ker h = \{\alpha \in \overset{k}{\Lambda}_r \mid (j_r s)^*\alpha = 0 \text{ for every section } s : \boldsymbol{X} \to \boldsymbol{Y}\}\,;$$

this partly shows the connection of $\overset{k}{\Theta}_r$ with the calculus of variations [13, 15, 16, 31, 32, 33].

Standard arguments of homological algebra prove that the following diagram is commutative, and its rows and columns are exact.

$$\begin{array}{ccccccccccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \overset{1}{\Theta}_r & \xrightarrow{d} & \overset{2}{\Theta}_r & \xrightarrow{d} & \dots & \xrightarrow{d} & \overset{I}{\Theta}_r & \xrightarrow{d} & 0 & \longrightarrow & \dots & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & I\!\!R & \longrightarrow & \overset{0}{\Lambda}_r & \xrightarrow{d} & \overset{1}{\Lambda}_r & \xrightarrow{d} & \overset{2}{\Lambda}_r & \xrightarrow{d} & \dots & \xrightarrow{d} & \overset{I}{\Lambda}_r & \xrightarrow{d} & \overset{I+1}{\Lambda}_r & \xrightarrow{d} & \dots & \xrightarrow{d} & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & I\!\!R & \longrightarrow & \overset{0}{\Lambda}_r & \xrightarrow{\mathcal{E}_0} & \overset{1}{\Lambda}_r/\overset{1}{\Theta}_r & \xrightarrow{\mathcal{E}_1} & \overset{2}{\Lambda}_r/\overset{2}{\Theta}_r & \xrightarrow{\mathcal{E}_2} & \dots & \xrightarrow{\mathcal{E}_{I-1}} & \overset{I}{\Lambda}_r/\overset{I}{\Theta}_r & \xrightarrow{\mathcal{E}_I} & \overset{I+1}{\Lambda}_r & \xrightarrow{d} & \dots & \xrightarrow{d} & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & 0 & & & & 0 & & 0 & & & &
\end{array}$$

Definition 2.1. The above diagram is said to be the *r–th order variational bicomplex* associated with the fibred manifold $\boldsymbol{Y} \to \boldsymbol{X}$ We say the bottom row of the above diagram to be the r–th order *variational sequence* associated with the fibred manifold $\boldsymbol{Y} \to \boldsymbol{X}$ [13]. □

Remark 2.1. Let $s \leq r$. Then we have the injective sheaf morphism (see [13])

$$\chi^r_s : \left(\overset{k}{\Lambda}_s / \overset{k}{\Theta}_s\right) \to \left(\overset{k}{\Lambda}_r / \overset{k}{\Theta}_r\right) : [\alpha] \mapsto [\pi^{r*}_s \alpha] \ .$$

Hence, there is an inclusion of the s–th order variational bicomplex into the r–th order variational bicomplex. □

Remark 2.2. The main task of the paper is to find suitable sheaves of fibred morphisms that are naturally isomorphic to the quotient sheaves of the variational sequence. In particular, we restrict our analysis to the following *short variational sequence*

$$0 \longrightarrow I\!\!R \longrightarrow \overset{0}{\Lambda}_r \xrightarrow{\mathcal{E}_0} \overset{1}{\Lambda}_r / \overset{1}{\Theta}_r \xrightarrow{\mathcal{E}_1} \dots$$

$$\dots \xrightarrow{\mathcal{E}_n} \overset{n+1}{\Lambda}_r / \overset{n+1}{\Theta}_r \xrightarrow{\mathcal{E}_{n+1}} \mathcal{E}_{n+1}\left(\overset{n+1}{\Lambda}_r / \overset{n+1}{\Theta}_r\right) \xrightarrow{\mathcal{E}_{n+2}} 0$$

due to the fact that, to our knowledge, if $k \geq n+3$, there is no interpretation of the k–th column of the variational bicomplex in terms of geometric objects of the variational calculus. □

We will start the programme of the above remark with the sheaves of degree k, with $0 \leq k \leq n$. By recalling proposition 2.1 and the identity $\overset{k}{\Theta}_r = \ker h$ for $0 \leq k \leq n$ we have the following result.

Proposition 2.3. *Let $0 \leq k \leq n$. Then, the sheaf morphism h yields the isomorphism*

$$I_k : \overset{k}{\Lambda}_r / \overset{k}{\Theta}_r \to \overset{k}{\mathcal{H}}{}^h_{r+1} : [\alpha] \mapsto h(\alpha) \, .$$

Definition 2.2. Let us set $\overset{k}{\mathcal{V}}_r := \overset{k}{\mathcal{H}}{}^h_{r+1}$.

We say a section $L \in \overset{n}{\mathcal{V}}_r$ to be a *r–th order generalised Lagrangian.* □

It is worth to point out the inclusions

$$\overset{n}{\mathcal{H}}_r \subset \overset{n}{\mathcal{V}}_r \subset \overset{n}{\mathcal{H}}_{r+1}\,.$$

The sheaf of the r–th order Lagrangians of the standard literature (see, for example, [9, 10, 12, 24]) is $\overset{n}{\mathcal{H}}_r$. So, due to the above inclusion, an r–th order generalised Lagrangian can be regarded as a special kind of $r+1$–th order standard Lagrangian.

3 Euler–Lagrange morphism

In this section we show that the quotient sheaf $\overset{n+1}{\Lambda}_r/\overset{n+1}{\Theta}_r$ of the variational sequence is isomorphic to a certain subsheaf of the sheaf of sections of a vector bundle, namely a sheaf of Euler–Lagrange morphism of special type. By means of this isomorphism, we find that the sheaf morphism $\mathcal{E}_n$ coincides with the standard Euler–Lagrange operator.

It is possible to introduce a first simplification of the quotient sheaves. Next proposition provides a new quotient sheaf which is isomorphic to $\overset{k}{\Lambda}_r/\overset{k}{\Theta}_r$ but is made with 'smaller' sheaves.

Proposition 3.1. *Let $k > n$. Then, the projection h induces the natural sheaf isomorphism*

$$\left(\overset{k}{\Lambda}_r/\overset{k}{\Theta}_r\right) \to \left(\overset{k-n}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}\right)/h(d\ker h) : [\alpha] \mapsto [h(\alpha)]\ ,$$

where $h(d\ker h) = h(\overset{k}{\Theta}_r)$.

Note that the sheaf injection χ^r_s can be read through the above isomorphism. Moreover, we have the inclusions

$$h(d\ker h) \subset d_h(\overset{k-n}{\mathcal{C}}_r \wedge \overset{n-1}{\mathcal{H}}{}^h_{r+1}) \subset h(\overset{k}{\Theta}_{r+1}) = h(d\ker h)\,. \tag{3}$$

In order to find an isomorphism of $(\overset{k-n}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1})\ /h(d\ker h)$ with a sheaf of forms on a jet bundle we use a result by Kolář [12]. To proceed further, we need some notation. On any coordinate open subset $\boldsymbol{U} \subset \boldsymbol{Y}$ we set

$$\omega := \sum_{\lambda_1,\dots,\lambda_n=1}^{n} d^{\lambda_1}\wedge\dots\wedge d^{\lambda_n} = n!d^1\wedge\dots\wedge d^n\,, \qquad \omega_\lambda := i_{\partial_\lambda}\omega\,.$$

If $f \in (\overset{0}{\Lambda}_r)_{\boldsymbol{U}}$, then we set by induction

$$J_\lambda f := (д_{r+1})_\lambda f \,, \qquad J_{\underline{p}+\lambda} f := J_\lambda J_{\underline{p}} f \,;$$

analogously, we denote by $L_{J_{\underline{p}}}$ the iterated Lie derivative. We have $J_{\underline{p}} f \circ j_{r+|\underline{p}|} s = \partial_{\underline{p}}(f \circ j_r s)$. A Leibnitz' rule holds for $J_{\underline{p}}$ (see [24]); if $g \in \left(\overset{0}{\Lambda}_r\right)_{\boldsymbol{U}}$, then we have

$$J_{\underline{p}}(fg) = \sum_{\underline{q}+\underline{t}=\underline{p}} \frac{\underline{p}!}{\underline{q}!\underline{t}!} J_{\underline{q}} f \, J_{\underline{t}} g \,.$$

If a vertical vector field $u : \boldsymbol{Y} \to V\boldsymbol{Y}$ has the expression $u = u^i \partial_i$, then the prolongation $u_r : J_r\boldsymbol{Y} \to V J_r \boldsymbol{Y}$ has the expression $u_r = J_{\underline{p}} u^i \partial_i^{\underline{p}}$.

Theorem 3.1. *(First variation formula for higher order variational calculus [12]) Let $\alpha \in \overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}$. Then there is a unique pair of sheaf morphisms*

$$E_\alpha \in \overset{1}{\mathcal{C}}_{(2r,0)} \wedge \overset{n}{\mathcal{H}}{}^h_{2r+1} \,, \qquad F_\alpha \in \overset{1}{\mathcal{C}}_{(2r,r)} \wedge \overset{n}{\mathcal{H}}{}^h_{2r+1} \,,$$

such that

i. $(\pi^{2r+1}_{r+1})^* \alpha = E_\alpha - F_\alpha$;

ii. F_α *is locally of the form* $F_\alpha = d_h p_\alpha$, *with* $p_\alpha \in \overset{1}{\mathcal{C}}_{(2r-1,r-1)} \wedge \overset{n}{\mathcal{H}}_{2r}$.

The uniqueness of the decomposition in the above theorem implies that both E_α and F_α are intrinsic geometric objects. In general, it is possible to determine a global p_α fulfilling the above conditions, but there is not a uniquely determined (hence intrinsic p_α unless $\dim \boldsymbol{X} = 1$ or $r = 1, 2$ (see [12]).

In coordinates, if $\alpha = \alpha_i^{\underline{p}} \vartheta^i_{\underline{p}} \wedge \omega$, then we have the well–known expression

$$E_\alpha = (-1)^{|\underline{p}|} J_{\underline{p}} \alpha_i^{\underline{p}} \vartheta^i \wedge \omega \,, \qquad 0 \leq |\underline{p}| \leq r \,. \tag{4}$$

Proposition 3.2. *We have the injective sheaf morphism*

$$I_{n+1} : \left(\overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1} \right) / h(d \ker h) \to \overset{n+1}{\Lambda}{}_{2r+1} : [\alpha] \mapsto \alpha + d_h p_\alpha \,.$$

Proof. We make use of the injective morphism χ^s_r of remark 2.1, and the inclusions (3). The morphism I_{n+1} is well–defined. In fact, if $\alpha, \beta \in \overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}$ such that $\beta = \alpha + d_h q$, then

$$\beta + d_h p_\beta = \alpha + d_h q + d_h p_\alpha + d_h p_{d_h q} \,,$$

where $d_h p_{d_h q} = -d_h q$, due to the uniqueness.

We have to prove that the morphism is injective. Suppose that $\beta + d_h p_\beta = \alpha + d_h p_\alpha$. Then

$$\beta - \alpha = h(d_h(p_\alpha - p_\beta) + d_v(p_\alpha - p_\beta)) = h(d(p_\alpha - p_\beta)),$$

and $h(p_\alpha - p_\beta) = 0$ yields the result. QED

The final step is to characterise the image of I_{n+1}.

Theorem 3.2. *We have the sheaf isomorphism*

$$I_{n+1} : \overset{n+1}{\Lambda}_r / \overset{n+1}{\Theta}_r \to \overset{n+1}{\mathcal{V}}_r,$$

where $\overset{n+1}{\mathcal{V}}_r$ *is the sheaf generated by the presheaf*

$$\left(\overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1} + d_h(\overset{1}{\mathcal{C}}_{(2r,r-1)} \wedge \overset{n-1}{\mathcal{H}}_{2r})\right) \cap \left(\overset{1}{\mathcal{C}}_{(2r+1,0)} \wedge \overset{n}{\mathcal{H}}_{2r+1}\right) .$$

PROOF. It comes from the isomorphism of proposition 3.1, the injective morphism I_{n+1} and the characterisation of the image of I_{n+1} provided by theorem 3.1. QED

Remark 3.1. The sheaf $\overset{n+1}{\mathcal{V}}_r$ is a sheaf of $\mathbb{R}$–vector spaces, but it does not have the structure of a module over the ring $\mathcal{C}^\infty(J_{2r+1}\boldsymbol{Y})$; hence, it cannot be the sheaf of sections of some vector bundle over $J_{2r+1}\boldsymbol{Y}$ [36]. □

Now, we can evaluate $\mathcal{E}_n$ by means of the isomorphisms I_n, I_{n+1}.

Theorem 3.3. *Let* $\alpha \in \overset{n}{\mathcal{V}}_r$. *Then,* $\mathcal{E}_n(\alpha) \in \overset{n+1}{\mathcal{V}}_r$ *coincides with the standard higher order Euler–Lagrange morphism [9, 10, 12, 24] associated with the generalised* r*–th order Lagrangian* α*, regarded as a standard* $(r+1)$*–th order Lagrangian.*

PROOF. In fact, Theorem 3.1 yields the standard higher order Euler–Lagrange morphism. Moreover, we have the inclusions $\overset{n}{\mathcal{V}}_r \subset \overset{n}{\mathcal{H}}_{r+1} \subset \overset{n}{\mathcal{V}}_{r+1}$. The result now is immediate, due to the commutativity of the inclusion of the bicomplex of order r into the bicomplex of order $r+1$ (Remark 2.1). QED

Definition 3.1. Let $\alpha \in \overset{n+1}{\Lambda}_r$.

We say $E_{h(\alpha)} \in \overset{n+1}{\mathcal{V}}_r$ to be the *generalised* r*–th order Euler–Lagrange morphism* associated with α.

We say $p_{h(\alpha)}$ to be a *generalised* r*–th order momentum* associated with α.

We say $\mathcal{E}_n$ to be the *generalised* r*–th order Euler–Lagrange operator.*

Remark 3.2. It is worth to point out the inclusions

$$\overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}_r \subset \overset{n+1}{\mathcal{V}}_r \subset \overset{1}{\mathcal{C}}_{2r+1} \wedge \overset{n}{\mathcal{H}}_{2r+1} \,.$$

The sheaf of the r–th order Euler–Lagrange morphisms of the standard literature (see, for example, [9, 10, 12, 24]) is $\overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}_r$. So, due to the above inclusion, an r–th order generalised Euler–Lagrange morphism can be regarded as a special kind of $2r+1$–th order standard Euler–Lagrange morphism. □

Remark 3.3. It is interesting to note that, by using a non–horizontal Lagrangian in the geometric formulation of the action (see [5, 6, 7, 9, 20]), some theories which are based upon polynomial $(r+1)$–th order horizontal Lagrangians can be seen also as r–th order theories. □

4 Helmholtz morphism

It is known [5, 13] that there exists a locally defined geometric object, namely the Helmholtz morphism, whose vanishing is equivalent to the local conditions of local variationality [2, 1, 3, 11, 18, 13, 27]. We show that the Helmholtz morphism is *intrinsically characterised* by means of the Euler–Lagrange morphism. As a by–product, we obtain a new intrinsic geometrical object which plays a role analogous to the role of the momentum of a Lagrangian.

More precisely, we find an isomorphism of the sheaf

$$\mathcal{E}_{n+1}(\overset{n+1}{\mathcal{V}}_r) \simeq \mathcal{E}_{n+1}(\overset{n+1}{\Lambda}_r / \overset{n+1}{\Theta}_r) = d\overset{n+1}{\Lambda}_r / d\overset{n+1}{\Theta}_r \,.$$

with a subsheaf of a sheaf of sections of a vector bundle. Hence, we will be able to provide an explicit expression for the map $\mathcal{E}_{n+1}$. We reduce our search by using the results of the above section.

Lemma 4.1. *We have the natural injection*

$$d\overset{n+1}{\Lambda}_r / d\overset{n+1}{\Theta}_r \to \left(\overset{2}{\mathcal{C}}_{2r+1} \wedge \overset{n}{\mathcal{H}}{}^h_{2r+2} \right) / h(d \ker h) : [d\alpha] \mapsto [dE_{h(\alpha)}] \,.$$

PROOF. It is a direct consequence of $dd_h = -d_h d_v$ and

$$\alpha = E_{h(\alpha)} - d_h p_{h(\alpha)} + v(\alpha) \,. \quad \boxed{QED}$$

Lemma 4.2. *Let* $\beta \in \overset{1}{\mathcal{C}}_s \wedge \overset{1}{\mathcal{C}}_{(s,0)} \wedge \overset{n}{\mathcal{H}}_s$*. Then, there is a unique*

$$\tilde{H}_\beta \in \overset{1}{\mathcal{C}}_{(2s,s)} \otimes \overset{1}{\mathcal{C}}_{(2s,0)} \wedge \overset{n}{\mathcal{H}}_{2s}$$

such that, for all $u : \boldsymbol{Y} \to V\boldsymbol{Y}$*,*

$$E_{\hat{\beta}} = C^1_1 \left(u_{2s} \otimes \tilde{H}_\beta \right) ,$$

where $\hat{\beta} := i_{u_s}\beta$*, and* C^1_1 *stands for tensor contraction.*

Proof. Let $\boldsymbol{U} \subset \boldsymbol{Y}$ be an open coordinate subset, and suppose that we have the expression on $\boldsymbol{U}$

$$\beta = \beta^{\underline{p}}_{i\,j} \vartheta^i_{\underline{p}} \wedge \vartheta^j \wedge \omega \,, \qquad 0 \leq |\underline{p}| \leq s \,.$$

Then we have the coordinate expression

$$E_{\hat{\beta}} = J_{\underline{p}} u^i \left(\beta^{\underline{p}}_{i\,j} - \sum_{|\underline{q}|=0}^{s-|\underline{p}|} (-1)^{|\underline{p}+\underline{q}|} \frac{(\underline{p}+\underline{q})!}{\underline{p}!\underline{q}!} J_{\underline{q}} \beta^{\underline{p}+\underline{q}}_{j\ \ i} \right) \vartheta^j \wedge \omega \,.$$

Let us set

$$\tilde{H}_\beta[\boldsymbol{U}] := \left(\beta^{\underline{p}}_{i\,j} - \sum_{|\underline{q}|=0}^{s-|\underline{p}|} (-1)^{|\underline{p}+\underline{q}|} \frac{(\underline{p}+\underline{q})!}{\underline{p}!\underline{q}!} J_{\underline{q}} \beta^{\underline{p}+\underline{q}}_{j\ \ i} \right) \vartheta^i_{\underline{p}} \otimes \vartheta^j \wedge \omega \,.$$

Then, by the arbitrariness of u, $\tilde{H}_\beta[\boldsymbol{U}]$ is the unique morphism fulfilling the conditions of the statement on $\boldsymbol{U}$.

If $\boldsymbol{V} \subset \boldsymbol{Y}$ is another open coordinate subset and $\boldsymbol{U} \cap \boldsymbol{V} \neq \emptyset$, then, by uniqueness, we have $\tilde{H}_\beta[\boldsymbol{U}]|_{\boldsymbol{U}\cap\boldsymbol{V}} = \tilde{H}_\beta[\boldsymbol{V}]|_{\boldsymbol{U}\cap\boldsymbol{V}}$. Hence, we obtain the result by setting $\tilde{H}_\beta|_{\boldsymbol{U}} := \tilde{H}_\beta[\boldsymbol{U}]$ on any coordinate open subset $\boldsymbol{U} \subset \boldsymbol{Y}$. QED

Theorem 4.1. *(Generalised second variation formula).*
Let $\beta \in \overset{1}{\mathcal{C}}_s \wedge \overset{1}{\mathcal{C}}_{(s,0)} \wedge \overset{n}{\mathcal{H}}_s$*. Then, there is a unique pair of sheaf morphisms*

$$H_\beta \in \overset{1}{\mathcal{C}}_{(2s,s)} \wedge \overset{1}{\mathcal{C}}_{(2s,0)} \wedge \overset{n}{\mathcal{H}}_{2s} \,, \quad G_\beta \in \overset{2}{\mathcal{C}}_{(2s,s)} \wedge \overset{n}{\mathcal{H}}_{2s} \,,$$

such that

i. $\pi^{2s\,*}_s \beta = H_\beta - G_\beta$

ii. $H_\beta = 1/2\, A(\tilde{H}_\beta)$*, where* A *is the antisymmetrisation map.*

Moreover, G_β *is locally of the type* $G_\beta = d_h q_\beta$*, where* $q_\beta \in \overset{2}{\mathcal{C}}_{2s-1} \wedge \overset{n-1}{\mathcal{H}}_{2s-1}$*, hence* $[\beta] = [H_\beta]$*.*

Proof. It is clear that G_β is uniquely determined by β and the choice $H_\beta = 1/2\, A(\tilde{H}_\beta)$.

Moreover, it can be easily seen [24] by induction on $|\underline{p}|$ that, on a coordinate open subset $\boldsymbol{U} \subset \boldsymbol{Y}$, we have

$$\beta = \beta^{\underline{p}}_{i\,j} \vartheta^i_{\underline{p}} \wedge \vartheta^j \wedge \omega = \beta^{\underline{p}}_{i\,j} L_{\underline{p}}(\vartheta^i) \wedge \vartheta^j \wedge \omega = (-1)^{|\underline{p}|} \vartheta^i \wedge L_{\underline{p}}(\beta^{\underline{p}}_{i\,j} \vartheta^j) \wedge \omega + 2 d_h q_\beta \,,$$

which yields the thesis by the Leibnitz' rule, the injective morphism χ^s_r of remark 2.1, and the inclusions (3) (a similar result can be found in [5, 13]). QED

Remark 4.1. In general, the section q_β is not uniquely characterised. But, if $\dim \boldsymbol{X} = 1$, then there exists a unique q_β fulfilling the conditions of the statement of the above theorem. □

Corollary 4.1. *The sheaf* $\mathcal{E}_{n+1} \left(\overset{n+1}{\mathcal{V}}_r \right)$ *is isomorphic to the image of the injective morphism*

$$I_{n+2} : d \overset{n+1}{\Lambda}_r / d \overset{n+1}{\Theta}_r \to \overset{1}{\mathcal{C}}_{4r+1} \wedge \overset{1}{\mathcal{C}}_{(4r+1,0)} \wedge \overset{n}{\mathcal{H}}_{4r+1} : [d\alpha] \mapsto H_{dE_{h(\alpha)}} \,.$$

Proof. I_{n+2} is well defined due to the uniqueness of the decomposition of the generalised second variation formula. Moreover, being $E_{h(\alpha)}$ affine with respect to the highest order derivatives, I_{n+2} is valued into

$$\overset{1}{\mathcal{C}}_{4r+1} \wedge \overset{1}{\mathcal{C}}_{(4r+1,0)} \wedge \overset{n}{\mathcal{H}}_{4r+1} \subset \overset{1}{\mathcal{C}}_{4r+2} \wedge \overset{1}{\mathcal{C}}_{(4r+2,0)} \wedge \overset{n}{\mathcal{H}}_{4r+2}$$

The injectivity of I_{n+2} follows from the above theorem, because if $dE_{h(\alpha)}$ and $dE_{h(\beta)}$ fulfill $H_{dE_{h(\alpha)}} = H_{dE_{h(\beta)}}$, then we have (locally)

$$dE_{h(\alpha)} - dE_{h(\beta)} = d_h(q_{h(\alpha)} - q_{h(\beta)}) \,.$$ QED

Corollary 4.2. *The sheaf morphism* $\mathcal{E}_{n+1}$ *can be expressed via* I_{n+1} *and* I_{n+2} *by*

$$\mathcal{E}_{n+1} : \overset{n+1}{\mathcal{V}}_r \to \overset{1}{\mathcal{C}}_{4r+1} \wedge \overset{1}{\mathcal{C}}_{(4r+1,0)} \wedge \overset{n}{\mathcal{H}}_{4r+1} : E \mapsto H_{dE} \,.$$

Moreover, if the coordinate expression of E *is* $E = E_j \vartheta^j \wedge \omega$*, then the coordinate expression of* $\mathcal{E}_{n+1}(E)$ *is*

$$\mathcal{E}_{n+1}(E) \;= \frac{1}{2} \left(\partial^{\underline{p}}_i E_j - \sum_{|\underline{q}|=0}^{2r+1-|\underline{p}|} (-1)^{|\underline{p}+\underline{q}|} \frac{(\underline{p}+\underline{q})!}{\underline{p}!\underline{q}!} J_{\underline{q}} \partial^{\underline{p}+\underline{q}}_j E_i \right) \vartheta^i_{\underline{p}} \wedge \vartheta^j \wedge \omega \,.$$

Definition 4.1. Let $\alpha \in \overset{n+1}{\Lambda}_r$.

We say $H_{dE_{h(\alpha)}}$ to be the *generalised r–th order Helmholtz morphism.*

We say $q_{dE_{h(\alpha)}}$ to be a *generalised r–th order momentum* associated to the Helmholtz morphism.

We say $\mathcal{E}_{n+1}$ to be the *generalised r–th order Helmholtz operator.* □

5 Inverse problems

In this section, we show that the results of the above sections toghether with the exactness of the variational sequence yield the solution for two important inverse problems: the minimal order variationally trivial Lagrangians and the minimal order Lagrangian corresponding to a locally variational Euler–Lagrange morphism.

We can summarise the results of the above sections in the following theorem.

Theorem 5.1. *The r–th order short variational sequence is isomorphic to the exact sequence*

$$0 \longrightarrow I\!R \longrightarrow \overset{0}{\Lambda}_r \xrightarrow{\mathcal{E}_0} \overset{1}{\mathcal{V}}_r \xrightarrow{\mathcal{E}_1} \dots$$

$$\dots \xrightarrow{\mathcal{E}_{n-1}} \overset{n}{\mathcal{V}}_r \xrightarrow{\mathcal{E}_n} \overset{n+1}{\mathcal{V}}_r \xrightarrow{\mathcal{E}_{n+1}} \mathcal{E}_{n+1}\left(\overset{n+1}{\mathcal{V}}_r\right) \xrightarrow{\mathcal{E}_{n+2}} 0,$$

We have two main consequences of the exactness of the above sequence.

Corollary 5.1. *Let $L \in (\overset{n}{\mathcal{V}}_r)_{\boldsymbol{Y}}$ such that $\mathcal{E}_n(L) = 0$. Then, for any $y \in \boldsymbol{Y}$ there exist an open neighbourhood $\boldsymbol{U} \subset \boldsymbol{Y}$ of y and a section $T \in (\overset{n-1}{\mathcal{V}}_r)_{\boldsymbol{U}}$ such that $\mathcal{E}_{n-1}(T) = L$. If $H^n_{de\ Rham}\boldsymbol{Y} = 0$, then we can choose $\boldsymbol{U} = \boldsymbol{Y}$.*

PROOF. The first statement comes from the definition of exactness for a sheaf sequence. The second statement comes from the abstract de Rham theorem; in fact, the (long) variational sequence is a (soft) resolution of the constant sheaf $I\!R$ (see [13, 36]). QED

Definition 5.1. Let $L \in (\overset{n}{\mathcal{V}}_r)_{\boldsymbol{Y}}$ such that $\mathcal{E}_n(L) = 0$. We say L to be a variationally trivial r–th order (generalised) Lagrangian.

□

Remark 5.1. If $L \in \overset{n}{\mathcal{V}}_r$ is variationaly trivial, then L is (locally) of the form $L = \mathcal{E}_{n-1}(h(\alpha)) = d_h\alpha$, with $\alpha \in \overset{n-1}{\Lambda}_r$.

We stress that a similar result is obtained in [14], but with a computational proof. □

As for $(\overset{n+1}{\mathcal{V}}_r)_{\boldsymbol{Y}}$, we have a result which is analogous to the above corollary, and justifies the following definition.

Definition 5.2. Let $E \in (\overset{n+1}{\mathcal{V}}_r)_{\boldsymbol{Y}}$. If $\mathcal{E}_{n+1}(E) = 0$, then we say E to be a locally variational (generalised) r–th order Euler–Lagrange morphism. □

So, to any locally variational Euler–Lagrange morphism there exists a local Lagrangian whose associated Euler–Lagrange morphism (locally) coincides with the given one. This is a well–known fact in the theory of infinite order Lagrangian bicomplexes, but the novelty provided by our approach is the *minimality* of the order of the local Lagrangian. In fact, we have the following obvious proposition.

Proposition 5.1. *Let $E \in (\overset{n+1}{\mathcal{V}}_r)_{\boldsymbol{Y}}$ such that $E \notin (\overset{n+1}{\mathcal{V}}_{r-1})_{\boldsymbol{Y}}$. Let E be locally variational. Then, for any (local) Lagrangian $L \in \overset{n}{\mathcal{V}}_r$ of E, we have $L \notin \overset{n}{\mathcal{V}}_{r-1}$.*

Remark 5.2. In the literature there are similar results [2, 1, 4], but proofs are done by computations. The finite order variational sequence provides a structural answer to the minimal order Lagrangian problem.

Remark 5.3. We stress that a minimal order Lagrangian $L \in \overset{n}{\mathcal{V}}_r$ for a locally variational Euler–Lagrange morphism $E \in \overset{n+1}{\mathcal{V}}_r$ can be *explicitely* computed.

Namely, we pick an $\alpha \in \overset{n+2}{\Lambda}_r$ corresponding to the Euler–Lagrange morphism (i.e. , $I_{n+1}(h(\alpha)) = E$), and apply the contact homotopy operator (which is just the restriction of the Poincaré's homotopy operator to $\overset{n+2}{\Theta}_r$) to the closed form $d\alpha \in \overset{n+2}{\Theta}_r$, finding $\beta \in \overset{n+1}{\Theta}_r$ such that $d\beta = d\alpha$. By using once again using the (standard) homotopy operator we find $\gamma \in \overset{n}{\Lambda}_r$ such that $d\gamma = \beta - \alpha : L := I_n(\gamma)$ is the minimal order Lagrangian.

We recall that the well–known Volterra–Vainberg method for finding a Lagrangian for E yields a $(2r + 1)$–th order Lagrangian.

□

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