Classification of bi-Hamiltonian pairs extended by isometries

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Dedicated to Allan Fordy
on the occasion of his 70th birthday

Abstract

The aim of this article is to classify pairs of first-order Hamiltonian operators of Dubrovin-Novikov type such that one of them has a non-local part defined by an isometry of its leading coefficient. An example of such bi-Hamiltonian pair was recently found for the constant astigmatism equation. We obtain a classification in the case of 2 dependent variables, and a significant new example with 3 dependent variables that is an extension of a hydrodynamic type system obtained from a particular solution of the WDVV equations.

1 Introduction

The theory of homogeneous first order differential-geometric Poisson brackets was established by B.A. Dubrovin and S.P. Novikov in 1983 [3] in the framework of the Hamiltonian formalism for PDEs. Such Poisson brackets are defined by local differential operators $A = (A^{ij})$ of the type

$$A^{ij} = g^{ij}(\mathbf{u})\partial_x + \Gamma^{ij}_k(\mathbf{u})u^k_x, \tag{1}$$

where $u^i = u^i(t,x)$ are field variables, $i = 1, \ldots, n$ depending on two independent variables t, x. The operator A yields a Poisson bracket between densities

$$\{F,G\}_A = \int \frac{\delta F}{\delta u^i} A^{ij} \frac{\delta G}{\delta u^j}\, dx$$

if and only if, in the non degenerate case $\det(g^{ij}) \neq 0$, g^{ij} is symmetric, $\Gamma^{ij}_k = -g^{is}\Gamma^j_{sk}$ are Christoffel symbols of the Levi–Civita connection of g_{ij} (the inverse of g^{ij}), and the tensor $g^{ij}(\mathbf{u})$ is a flat contravariant metric.

The theory of compatible pairs of such Hamiltonian operators was developed later in a series of publications (see the review paper [10]). We recall that two Hamiltonian operators A, B, are said to be compatible if $A + \lambda B$ is a Hamiltonian operator for every $\lambda \in \mathbb{R}$. The main application of compatible pairs A_0, B_0 of homogeneous Hamiltonian operators of the type (1) is the integrability of the corresponding quasilinear systems of PDEs of the form

$$u^i_t = V^i_j(\mathbf{u})u^j_x,$$

which is provided by Magri's Theorem [8] when the above system of PDEs is *bi-Hamiltonian*:

$$u^i_t = V^i_j(\mathbf{u})u^j_x = A^{ij}_0 \frac{\delta \mathbf{H}_0}{\delta u^j} = B^{ij}_0 \frac{\delta \tilde{\mathbf{H}}_0}{\delta u^j}, \tag{2}$$

where $B^{ij}_0 = \tilde{g}^{ij}(\mathbf{u})\partial_x + \tilde{\Gamma}^{ij}_k(\mathbf{u})u^k_x$ and

$$\mathbf{H}_0 = \int h_0(\mathbf{u})dx, \quad \tilde{\mathbf{H}}_0 = \int \tilde{h}_0(\mathbf{u})dx,$$

where $h_0(\mathbf{u})$ and $\tilde{h}_0(\mathbf{u})$ are hydrodynamic conservation law densities.

In some cases, bi-Hamiltonian hydrodynamic type systems can be considered as the dispersionless limit of integrable bi-Hamiltonian systems, containing higher order derivatives:

$$u^i_t = A^{ij} \frac{\delta \mathbf{H}}{\delta u^j} = B^{ij} \frac{\delta \tilde{\mathbf{H}}}{\delta u^j},$$

where

$$\mathbf{H} = \int h(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, ...)dx, \quad \tilde{\mathbf{H}} = \int \tilde{h}(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, ...)dx$$

and A^{ij}, B^{ij} are non-homogeneous differential operators of arbitrary orders. In the dispersionless limit ($\partial_x \to \epsilon\partial_x,\ \partial_t \to \epsilon\partial_t,\ \epsilon \to 0$) we have $A^{ij} \to A_0^{ij}, B^{ij} \to B_0^{ij}$ and $h(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \ldots) \to h(\mathbf{u})$, $\tilde{h}(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \ldots) \to \tilde{h}(\mathbf{u})$.

A large number of examples of the above bi-Hamiltonian systems are known. They can be generated from solutions of the WDVV equations as Frobenius manifolds, for example [2].

In our paper, we consider an alternative class of bi-Hamiltonian systems. According with an observation made by B.A. Dubrovin and S.P. Novikov, one can introduce the so-called flat coordinates $a^k(\mathbf{u})$ such that the first local Hamiltonian operator A_0^{ij} takes the constant form, i.e. $A_0^{ij} = \eta^{ij}\partial_x$, where η^{ij} is a symmetric constant non degenerate matrix. In this case, the flat coordinates a^k play the role of "Liouville coordinates" for the second local Hamiltonian operator B_0^{ij}, *i.e.* we have $B_0^{ij} = \tilde{\Gamma}^{ji}\partial_x + \partial_x\tilde{\Gamma}^{ij}$, where $\tilde{\Gamma}^{ij}$ are a set of functions such that the second metric tensor is given by $\tilde{g}^{ij}(\mathbf{a}) = \tilde{\Gamma}^{ij}(\mathbf{a}) + \tilde{\Gamma}^{ji}(\mathbf{a})$, and $(\tilde{\Gamma}^{ij})_{,k} \equiv \tilde{\Gamma}^{ij}_k = -\tilde{g}^{is}\tilde{\Gamma}^j_{sk}$. Here we denote $(\tilde{\Gamma}^{ij})_{,k} \equiv \partial\tilde{\Gamma}^{ij}/\partial a^k$. So, we can rewrite the bi-Hamiltonian hydrodynamic type system (2) as

$$a^i_t = \eta^{ij}\partial_x\frac{\delta\mathbf{H}}{\delta a^j} = (\tilde{\Gamma}^{ji}\partial_x + \partial_x\tilde{\Gamma}^{ij})\frac{\delta\tilde{\mathbf{H}}}{\delta a^j}.$$

It is well-known that the class of local first-order homogeneous Hamiltonian operators can be extended to include non-local terms. A widely studied extension is that of Mokhov–Ferapontov and Ferapontov operators (see [5] and references therein), which is again homogeneous. Later E.V. Ferapontov introduced and studied (see [4]) a *non-homogeneous, non-local* extension of the form

$$B = \tilde{g}^{ij}\partial_x + \tilde{\Gamma}^{ij}_k u^k_x + \epsilon f^i\partial_x^{-1}f^j, \tag{3}$$

where ϵ is a parameter and (f^j) is a vector field, $f^j = f^j(\mathbf{u})$, that is an infinitesimal isometry of $\tilde{g}^{ij}$. All these Hamiltonian operators are applicable for integrable as well as for non-integrable systems. In our paper we deal with bi-Hamiltonian structures, which is a significant part in theory of integrable systems. Our motivating example is given by the constant astigmatism equation

$$u_{tt} + \left(\frac{1}{u}\right)_{xx} + 2 = 0, \tag{4}$$

whose bi-Hamiltonian structure, after introducing the variable $u_t = v_x$ and rewriting the equation as the non-homogeneous quasilinear system

$$u_t = v_x, \qquad v_t = -\left(\frac{1}{u}\right)_x - 2x, \tag{5}$$

was found in [14] to be

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x, \tag{6}$$

$$B = \begin{pmatrix} 2u & 0 \\ 0 & \frac{2}{u} \end{pmatrix} \partial_x + \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{u^2} \end{pmatrix} u_x + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} v_x + \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \partial_x^{-1}. \tag{7}$$

In this paper we investigate the following problem: *find all bi-Hamiltonian pairs A, B where*

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x, \quad B^{ij} = \Gamma^{ji}\partial_x + \partial_x \Gamma^{ij} + \epsilon f^i \partial_x^{-1} f^j, \tag{8}$$

where (f^i) is an infinitesimal isometry of $g^{ij} = \Gamma^{ji} + \Gamma^{ij}$ (here ϵ is an arbitrary parameter). The classification is made with respect to the action of the group of local diffeomorphisms of the dependent variables. We observe that isometries of the leading coefficients have been used in a different classification problem of pairs of compatible *local* first-order homogeneous operators A_0, B_0 in [12].

We give a complete solution to the above problem in the case of $n = 2$ dependent variables. The solution of the system of conditions in the case $n = 3$ is more complicated and will be dealt with in the future. However, we provide here an interesting new example: the "isometric" extension of a hydrodynamic type system obtained as a solution of the WDVV equations.

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2 Preliminaries

Let us consider the homogeneous operator of the first order:

$$A_0^{ij} = g^{ij}\partial_x + \Gamma^{ij}_k u^k_x. \tag{9}$$

It is well-known [3] that the conditions necessary and sufficient for A_0 to be Hamiltonian are, in the non degenerate case $\det(g^{ij}) \neq 0$[1], that g^{ij} is symmetric, its inverse (g_{ij}) is a flat pseudo-Riemannian metric and $\Gamma^i_{jk} = -g_{jp}\Gamma^{ip}_k$ are the Christoffel symbols of the Levi-Civita connection of g_{ij}.

Let us consider two first-order homogeneous Hamiltonian operators A_0 and $B_0 = \tilde{g}^{ij}\partial_x + \tilde{\Gamma}^{ij}_k u^k_x$. The pair of metrics g and $\tilde{g}$ is said to be *almost compatible* if for every linear combination $g_\lambda := g + \lambda\tilde{g}$ we have

$$\Gamma^{ij}_{\lambda,k} = \Gamma^{ij}_k + \lambda\tilde{\Gamma}^{ij}_k; \tag{10}$$

the pair of metrics g and $\tilde{g}$ is said to be *compatible* if and only if, in addition to (10), the Riemann curvature tensor R_λ of the metric $g + \lambda\tilde{g}$ splits as the sum of the Riemann curvature tensors R of g and $\tilde{R}$ of $\tilde{g}$:

$$R^{ij}_{\lambda,kl} = R^{ij}_{kl} + \lambda\tilde{R}^{ij}_{kl}.$$

It can be proved that the Hamiltonian operators A_0, B_0 are compatible, *i.e.* $A_0 + \lambda B_0$ is a Hamiltonian operator for every λ, if and only if the corresponding metrics are compatible [11]. In this case we say A_0, B_0 to be a *bi-Hamiltonian pair*.

Now, let us consider a non-local non-homogeneous operator of the form

$$B^{ij} = g^{ij}\partial_x + \Gamma^{ij}_k u^k_x + c u^i_x \partial_x^{-1} u^j_x + \epsilon f^i \partial_x^{-1} f^j,$$

where c, ϵ are constants and (f^i) is a vector field, $f^i = f^i(\mathbf{u})$. In [4] it is shown that B defines a Poisson bracket if and only if the following conditions are satisfied:

1. g_{ij} is a pseudo-Riemannian metric and g_{ij} is compatible with the connection with Christoffel symbols $\Gamma^i_{jk} = -g_{jp}\Gamma^{ip}_k$;
2. the connection Γ^i_{jk} is symmetric and it has constant curvature c;

[1]From now on all operators are assumed to have a non degenerate leading term.

3. (f^i) is an infinitesimal isometry of g_{ij}, or, equivalently, $\nabla^i f^j + \nabla^j f^i = 0$;

4. the cyclic condition

$$f^j \nabla^i f^k + < \text{cyclic} >= 0$$

is fulfilled.

It is clear that the last condition is trivially satisfied for 2-dimensional spaces.

Remark 1. *We observe that if we require $c = 0$ then the operator B is of the form*

$$B = B_0 + \epsilon f^i \partial_x^{-1} f^j,$$

where B_0 is a local homogeneous first-order Hamiltonian operator.

Motivated by the example of the constant astigmatism equation (4), in the two-component case ($n = 2$) we consider pairs of Hamiltonian operators A, B, where

$$A = \eta^{ij} \partial_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x, \tag{11a}$$

$$B^{ij} = B_0 + f^i \partial_x^{-1} f^j = g^{ij} \partial_x + \Gamma^{ij}_k u^k_x + f^i \partial_x^{-1} f^j. \tag{11b}$$

The compatibility conditions of the above operators are given in the following theorem.

Theorem 2. *The above Hamiltonian operators A, B form a bi-Hamiltonian pair if and only if*

1. *A is compatible with B_0;*

2. *(f^i) is an isometry of the leading terms of both operators A and B.*

Proof. Indeed, A is clearly a Hamiltonian operator. Moreover, the operator

$$\lambda A + B = (\lambda \eta^{ij} + g^{ij}) \partial_x + \Gamma^{ij}_k u^k_x + f^i \partial_x^{-1} f^j \tag{12}$$

should be Hamiltonian for every λ. This is equivalent to the requirement that the operator

$$(\lambda \eta^{ij} + g^{ij}) \partial_x + \Gamma^{ij}_k u^k_x \tag{13}$$

is Hamiltonian for every λ, which is the first condition of the above statement. The second condition comes from the fact that (f^i) must be an isometry of the leading metric coefficient $\lambda \eta^{ij} + g^{ij}$ for every $\lambda \in \mathbb{R}$. □

It is easy to realize that the Hamiltonian operators A and B_0 are compatible if and only if

$$\Gamma^{ij}_k = \Gamma^{ij}_{\lambda,k}, \tag{14a}$$

$$R^{ij}_{kl} = 0 = R^{ij}_{\lambda,kl}, \tag{14b}$$

where $g^{ij}_\lambda = g^{ij} + \lambda\eta^{ij}$ and Γ^{ij}_k, R^{ij}_{kl} are the Christoffel symbols and the Riemannian curvature tensor of g, respectively. These are the equations that will be used in order to produce a classification of the pairs A, B when $n = 2$.

3 The case $n = 2$: classification

In the case $n = 2$ we can provide a classification of the bi-Hamiltonian pairs A, B as in (8) or (11). We denote the dependent variables by u and v.

We stress that working with a pair of the form (8) means that we used flat coordinates of the first operator in order to obtain it in the form $A^{ij} = \eta^{ij}\partial_x$, where (η^{ij}) is a constant non degenerate symmetric matrix. Then, by linear transformations of the dependent variables, we further reduced (η^{ij}) to the 'antidiagonal identity' form in (8). The only remaining coordinate freedom consists in translations and scalings, and they will be used to reduce the number of parameters in the canonical forms.

An immediate observation is that the vector field $f = (f^i)$ must be a linear combination of the following isometries of η^{ij}:

1. $f_1 = \partial_u$,
2. $f_2 = \partial_v$,
3. $f_3 = u\partial_u - v\partial_v$,

so that $f = a_1 f_1 + a_2 f_2 + a_3 f_3$. If $a_3 \neq 0$, then by translating u and v (this will preserve η^{ij}) we can reduce to the case $f = u\partial_u - v\partial_v$. Otherwise, if $a_3 = 0$ then by complex scaling $u \mapsto cu$, $v \mapsto \frac{v}{c}$ we can transform the isometry f to $f = \partial_u + \partial_v$, or $f = \partial_u$. This shows that a complete classification of compatible pairs A and B (up to transformations preserving η) reduces to the three distint cases

1. $f = \partial_u$,

2. $f = \partial_u + \partial_v$,
3. $f = u\partial_u - v\partial_v$.

Our strategy is the following: we find the most general form of metric leading coefficient g of B for which the selected vector field f is an isometry, then we apply Theorem 2. We will check the compatibility of the local operators A and B_0 by the compatibility of the corresponding metric leading coefficients.

3.1 Case $f = \partial_u$

In this case the metric can be written as $g^{ij} = g^{ij}(v)$.

Theorem 3. *If $f = \partial_u$, the metric g^{ij} in the operator B is one of the following:*

$$g_1^{ij} = \begin{pmatrix} \frac{\alpha}{v} & \beta \\ \beta & v \end{pmatrix} \qquad \alpha \neq \beta^2,$$
$$g_2^{ij} = \begin{pmatrix} g^{11}(v) & g^{12}(v) \\ g^{12}(v) & 0 \end{pmatrix} \qquad g^{12}(v) \neq 0,$$
$$g_3^{ij} = \begin{pmatrix} 0 & \beta \\ \beta & v \end{pmatrix} \qquad \beta \neq 0,$$

where $g^{11}(v)$ and $g^{12}(v)$ are arbitrary functions.

Obviously, a similar statement holds for $f = \partial_v$ by simple change of variable.

3.2 Case $f = \partial_u + \partial_v$

The solutions of the conditions are presented in the following Theorem.

Theorem 4. *If $f = \partial_u + \partial_v$ then g^{ij} is one of the following*

$$g_4^{ij} = \begin{pmatrix} f(-u+v) & -f(-u+v)+\beta \\ -f(-u+v)+\beta & f(-u+v) \end{pmatrix} \qquad \beta \neq 0,$$

where $f = f(-u+v)$ is an arbitrary non-constant function;

$$g_5^{ij} = \begin{pmatrix} -u+v & \beta \\ \beta & 0 \end{pmatrix} \qquad \beta \neq 0,$$

$$g_6^{ij} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \qquad \alpha\gamma \neq \beta^2.$$

3.3 Case $f = u\partial_u - v\partial_v$

Here the coefficients of g^{ij} depend, in principle, on both variables u, v. Solving the conditions yields the following results.

Theorem 5. *If $f = u\partial_u - v\partial_v$ the metric g^{ij} is one of the following*

$$g_7^{ij} = \begin{pmatrix} \frac{\alpha uv+\epsilon}{v^2} & \beta \\ \beta & 0 \end{pmatrix} \qquad \beta \neq 0,$$

$$g_8^{ij} = \begin{pmatrix} \frac{u}{v} & \beta \\ \beta & \frac{\alpha v}{u} \end{pmatrix} \qquad \alpha \neq \beta^2,$$

$$g_9^{ij} = \begin{pmatrix} \frac{\alpha}{F} & \beta \\ \beta & F \end{pmatrix} \qquad \alpha \neq \beta^2,$$

with

$$F = \frac{\gamma uv + \epsilon + \sqrt{(\gamma^2 - 4\alpha)\, u^2 v^2 + 2\gamma\epsilon uv + \epsilon^2}}{2u^2}. \tag{15}$$

4 Hierarchies in 2 components

It is well-known that a bi-Hamiltonian pair A, B defines a sequence of Poisson commuting conserved quantities H_k by means of Magri's recursion [8]:

$$A^{ij}\frac{\delta H_{k+1}}{\delta u^j} = B^{ij}\frac{\delta H_k}{\delta u^j}. \tag{16}$$

where H_1, H_2 are densities of conservation laws for the bi-Hamiltonian system of PDEs associated with the pair. The first density is usually taken as a Casimir of one of the operators, *i.e.* a density that is in the kernel of one of the operators. The sequence of systems of PDEs

$$u^i_{t^k} = A^{ij}\frac{\delta H_k}{\delta u^j} \tag{17}$$

is then the *integrable hierarchy*; the proper integrable system is identified as the first non-trivial system in the hierarchy.

It is possible to give a more explicit construction of the integrable hierarchies generated by a bi-Hamiltonian pair using recursion operators defined as

$R = B \circ A^{-1}$. This implies that there is no more need to solve Magri's recursion. In our case it is possible to give an explicit form of recursion operators associated with the bi-Hamiltonian pairs that we found in our classification in the case $n = 2$. In the following Section we will show by means of an example how to construct the integrable system which is the starting element of the integrable hierarchy.

1. Case g_1^{ij}. We have (α is an arbitrary constant)

$$B = \begin{pmatrix} \frac{\alpha}{v}\partial_x - \frac{\alpha}{2v^2}v_x + \partial_x^{-1} & \beta\partial_x + \frac{1}{2}u_x \\ \beta\partial_x - \frac{1}{2}u_x & v\partial_x + \frac{1}{2}v_x \end{pmatrix}$$

and the recursion operator is

$$R = \begin{pmatrix} \beta\partial_x + \frac{1}{2}u_x & \frac{\alpha}{v}\partial_x - \frac{\alpha}{2v^2}v_x + \partial_x^{-1} \\ v\partial_x + \frac{1}{2}v_x & \beta\partial_x - \frac{1}{2}u_x \end{pmatrix} \partial_x^{-1}$$

2. Case g_2^{ij}. We have (functions $g^{11}(v)$ and $g^{12}(v)$ are arbitrary):

$$B = \begin{pmatrix} g^{11}(v)\partial_x + \frac{1}{2}\frac{d}{dv}g^{11}(v)v_x + \partial_x^{-1} & g^{12}(v)\partial_x + \frac{d}{dv}g^{12}(v)v_x \\ g^{12}(v)\partial_x & 0 \end{pmatrix}$$

then, the recursion operator is

$$R = \begin{pmatrix} g^{12}(v)\partial_x + \frac{d}{dv}g^{12}(v)v_x & g^{11}(v)\partial_x + \frac{1}{2}\frac{d}{dv}g^{11}(v)v_x + \partial_x^{-1} \\ 0 & g^{12}(v)\partial_x \end{pmatrix}$$

3. Case g_3^{ij}. We have

$$B = \begin{pmatrix} \partial_x^{-1} & \beta\partial_x + \frac{1}{2}u_x \\ \beta\partial_x - \frac{1}{2}u_x & v\partial_x + \frac{1}{2}v_x \end{pmatrix}$$

and the recursion operator

$$R = \begin{pmatrix} \beta\partial_x + \frac{1}{2}u_x & \partial_x^{-1} \\ v\partial_x + \frac{1}{2}v_x & \beta\partial_x - \frac{1}{2}u_x \end{pmatrix} \partial_x^{-1}$$

4. Case g_4^{ij} (the function $f(\gamma)$ is arbitrary). For simplicity, let us substitute $\gamma := -u + v$, then the operator is

$$B = \begin{pmatrix} f(\gamma)\partial_x + \frac{f'(\gamma)}{2}(v_x - u_x) + \partial_x^{-1} & (-f(\gamma) + c_1)\partial_x + \frac{f'(\gamma)}{2}(u_x - v_x) + \partial_x^{-1} \\ (-f(\gamma) + c_1)\partial_x + \frac{f'(\gamma)}{2}(u_x - v_x) + \partial_x^{-1} & f(\gamma)\partial_x + \frac{f(\gamma)}{2}(v_x - u_x) + \partial_x^{-1} \end{pmatrix}$$

and the recursion operator is

$$R = \begin{pmatrix} (-f(\gamma)+c_1)\partial_x + \frac{f'(\gamma)}{2}(u_x - v_x) + \partial_x^{-1} & f(\gamma)\partial_x + \frac{f'(\gamma)}{2}(v_x - u_x) + \partial_x^{-1} \\ f(\gamma)\partial_x + \frac{f(\gamma)}{2}(v_x - u_x) + \partial_x^{-1} & (-f(\gamma)+c_1)\partial_x + \frac{f'(\gamma)}{2}(u_x - v_x) + \partial_x^{-1} \end{pmatrix} \partial_x^{-1}$$

5. Case g_5^{ij}. The operator is

$$B = \begin{pmatrix} (-u+v)\partial_x - \frac{1}{2}u_x + \frac{1}{2}v_x + \partial_x^{-1} & \beta\partial_x + \frac{1}{2}v_x + \partial_x^{-1} \\ \beta\partial_x - \frac{1}{2}v_x + \partial_x^{-1} & \partial_x^{-1} \end{pmatrix}$$

and the recursion operator is

$$R = \begin{pmatrix} \beta\partial_x + \frac{1}{2}v_x + \partial_x^{-1} & (-u+v)\partial_x - \frac{1}{2}u_x + \frac{1}{2}v_x + \partial_x^{-1} \\ \partial_x^{-1} & \beta\partial_x - \frac{1}{2}v_x + \partial_x^{-1} \end{pmatrix} \partial_x^{-1}$$

6. Case g_6^{ij}. The operator is (β and γ are arbitrary constants)

$$B = \begin{pmatrix} \alpha\partial_x + \partial_x^{-1} & \beta\partial_x + \partial_x^{-1} \\ \beta\partial_x + \partial_x^{-1} & \gamma\partial_x + \partial_x^{-1} \end{pmatrix}$$

and the recursion operator is

$$R = \begin{pmatrix} \beta\partial_x + \partial_x^{-1} & \alpha\partial_x + \partial_x^{-1} \\ \gamma\partial_x + \partial_x^{-1} & \beta\partial_x + \partial_x^{-1} \end{pmatrix} \partial_x^{-1}$$

7. Case g_7^{ij}. We have (β and γ are arbitrary constants)

$$B = \begin{pmatrix} \frac{\alpha uv+\epsilon}{v^2}\partial_x + \frac{\alpha}{2v}u_x + \left(\frac{\alpha u}{2v^2} - \frac{\alpha uv+\epsilon}{v^3}\right) v_x + u\partial_x^{-1}u & \beta\partial_x - \frac{\alpha}{2v}v_x - u\partial_x^{-1}v \\ \beta\partial_x + \frac{\alpha}{2v}v_x - v\partial_x^{-1}u & v\partial_x^{-1}v \end{pmatrix}$$

and we obtain the recursion operator:

$$R = \begin{pmatrix} \beta\partial_x - \frac{\alpha}{2v}v_x - u\partial_x^{-1}v & \frac{\alpha uv+\epsilon}{v^2}\partial_x + \frac{\alpha}{2v}u_x + \left(\frac{\alpha u}{2v^2} - \frac{\alpha uv+\epsilon}{v^3}\right) v_x + u\partial_x^{-1}u \\ v\partial_x^{-1}v & \beta\partial_x + \frac{\alpha}{2v}v_x - v\partial_x^{-1}u \end{pmatrix} \partial_x^{-1}.$$

8. Case g_8^{ij}. Let us consider the following (c is an arbitrary constant):

$$B = \begin{pmatrix} \frac{u}{v}\partial_x + \frac{1}{2v}u_x - \frac{u}{2v^2}v_x + u\partial_x^{-1}u & \beta\partial_x + \frac{\alpha}{2u}u_x - \frac{1}{2v}v_x - u\partial_x^{-1}v \\ \beta\partial_x - \frac{\alpha}{2u}u_x + \frac{1}{2v}v_x - v\partial_x^{-1}u & \frac{\alpha v}{u}\partial_x - \frac{\alpha v}{2u^2}u_x + \frac{\alpha}{2u}v_x + v\partial_x^{-1}v \end{pmatrix}.$$

The recursion operator is

$$R = \begin{pmatrix} \beta\partial_x + \frac{\alpha}{2u}u_x - \frac{1}{2v}v_x - u\partial_x^{-1}v & \frac{u}{v}\partial_x + \frac{1}{2v}u_x - \frac{u}{2v^2}v_x + u\partial_x^{-1}u \\ \frac{\alpha v}{u}\partial_x - \frac{\alpha v}{2u^2}u_x + \frac{\alpha}{2u}v_x + v\partial_x^{-1}v & \beta\partial_x - \frac{\alpha}{2u}u_x + \frac{1}{2v}v_x - v\partial_x^{-1}u \end{pmatrix} \partial_x^{-1}.$$

9. Case g_9^{ij}. The expressions of the Christoffel symbols make the Hamiltonian operator and the recursion operator too big to be shown here.

5 The constant astigmatism equation

In this section we show how to construct an integrable system underlying one of the bi-Hamiltonian pairs that we found so far. The example contains the constant astigmatism equation as a particular case; the calculation scheme can be repeated for any of the bi-Hamiltonian pairs in our classification.

Now, let us consider the metric g_1^{ij} in Theorem 3 and change the variables according with the map $u \mapsto v$, $v \mapsto u$; we obtain

$$g_1^{ij} = \begin{pmatrix} u & \beta \\ \beta & \frac{\alpha}{u} \end{pmatrix} \quad \alpha \neq \beta^2, \qquad f^i = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{18}$$

and the Christoffel symbols of g_1 are:

$$\begin{gathered} \Gamma_1^{12} = \Gamma_2^{22} = \Gamma_2^{11} = \Gamma_1^{21} = 0 \\ \Gamma_2^{21} = -\Gamma_2^{12} = \Gamma_1^{11} = \frac{1}{2}; \\ \Gamma_1^{22} = -\frac{\alpha}{2u^2}; \end{gathered}$$

The operator B is:

$$B = \begin{pmatrix} u & \beta \\ \beta & \frac{\alpha}{u} \end{pmatrix} \partial_x + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{\alpha}{2u^2} \end{pmatrix} u_x + \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} v_x + \epsilon \begin{pmatrix} 0 & 0 \\ 0 & \partial_x^{-1} \end{pmatrix}$$

Let us apply B to the Casimir $-2v$ of A:

$$B \begin{pmatrix} 0 \\ -2 \end{pmatrix} = 0 + \begin{pmatrix} 0 \\ \frac{\alpha}{u^2} \end{pmatrix} u_x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} v_x - \epsilon \begin{pmatrix} 0 \\ 2x \end{pmatrix}$$

Now, the constant astigmatism equation

$$u_{tt} + \left(\frac{1}{u}\right)_{xx} + 2 = 0 \tag{19}$$

can be written in the following form:

$$\begin{cases} u_t = v_x \\ v_t = -\left(\frac{1}{u}\right)_x - 2x \end{cases}$$

and the associated Hamiltonian operator of Dubrovin-Novikov type is B where $\alpha = 1$ and $\beta = 0$. By substituting we obtain the following Christoffel symbols:

$$\Gamma^{21}_{2} = -\Gamma^{12}_{2} = \Gamma^{11}_{1} = \frac{1}{2}$$

$$\Gamma^{22}_{1} = -\frac{\alpha}{2u^2} = -\frac{1}{2u^2}$$

Therefore, we have

$$\begin{cases} u_t = v_x \\ v_x = -\left(-\frac{\alpha}{u^2}\right) u_x - \epsilon x = -\left(\frac{1}{u}\right)_x - \epsilon x \end{cases}$$

In particular, we obtained the equation (19) as the second flow of the hierarchy generated by the bi-Hamiltonian structure defined by A and B. The Hamiltonian operator B has the expression

$$B = \begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix} \partial_x + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2u^2} \end{pmatrix} u_x + \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} v_x + \begin{pmatrix} 0 & 0 \\ 0 & \partial_x^{-1} \end{pmatrix} \tag{20}$$

and, according with the computations in Section 4 and the change of variables at the beginning of this Section, we obtain the following recursion operator:

$$R = \begin{pmatrix} u\partial_x + \frac{1}{2}u_x & -\frac{1}{2}v_x \\ +\frac{1}{2}v_x & \frac{1}{u}\partial_x - \frac{u_x}{2u^2} + \partial_x^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x^{-1} =$$
$$= \begin{pmatrix} -\frac{1}{2}v_x & u\partial_x + \frac{1}{2}u_x \\ \frac{1}{u}\partial_x - \frac{u_x}{2u^2} + \partial_x^{-1} & +\frac{1}{2}v_x \end{pmatrix} \partial_x^{-1}.$$

The reader can also refer to [14]. Note that after a simple change of coordinates the operator B in (20) is exactly the operator (7) in [14].

6 An example in three components: WDVV equation

The equations of associativity, or Witten–Dijkgraaf–Verlinde–Verlinde equations [1] contain commuting hydrodynamic type systems

$$a^i_{t^k} = \eta^{im} \left(\frac{\partial^2 F}{\partial a^m \partial a^k} \right)_x .$$

Indeed, the compatibility conditions $(a^i_{t^k})_{t^n} = (a^i_{t^n})_{t^k}$ lead directly to the WDVV system

$$\frac{\partial^3 F}{\partial a^i \partial a^k \partial a^p}\eta^{pq}\frac{\partial^3 F}{\partial a^q \partial a^n \partial a^j} = \frac{\partial^3 F}{\partial a^i \partial a^n \partial a^p}\eta^{pq}\frac{\partial^3 F}{\partial a^q \partial a^k \partial a^j}.$$

The concept of Frobenius manifold [2] is based on the existence of a second local Hamiltonian structure for these commuting hydrodynamic type systems. In the three-component case, if $\eta^{ij} = \delta^{i,4-i}$, then

$$F = \frac{1}{2}u^2 w + \frac{1}{2}uv^2 + f(v,w),$$

where the function $f(v,w)$ solves a single third order nonlinear differential equation

$$f_{www} = f_{vvw}^2 - f_{vww}f_{vvv}.$$

A simple nontrival solution found by B.A. Dubrovin leads to the ansatz

$$f = -\frac{1}{16}v^4\gamma(w),$$

which implies the remarkable Chazy equation

$$\gamma''' = 6\gamma\gamma'' - 9\gamma'^2.$$

In the semi-simple case, the velocity matrices $\eta^{pq}\frac{\partial^3 F}{\partial a^q \partial a^k \partial a^j}$ are non-degenerate. So, a generic solution $\gamma(w)$ of the Chazy equation determines three distinct characteristic roots of the above velocity matrix. Precisely, according to the construction by B.A. Dubrovin, we have a pair of commuting hydrodynamic type systems

$$u_t = \left(-\frac{1}{4}v^3\gamma'(w)\right)_x, \quad v_t = \left(u - \frac{3}{4}v^2\gamma(w)\right)_x, \quad w_t = v_x,$$

$$u_y = \left(-\frac{1}{16}v^4\gamma''(w)\right)_x, \quad v_y = \left(-\frac{1}{4}v^3\gamma'(w)\right)_x, \quad w_y = u_x.$$

The corresponding velocity matrices

$$\eta^{pq}\frac{\partial^3 F}{\partial a^q \partial a^k \partial a^2} \quad \text{and} \quad \eta^{pq}\frac{\partial^3 F}{\partial a^q \partial a^k \partial a^3}$$

are non degenerate, with the exception of the particular case $\gamma(w) = -2/w$. In the latter case, the hydrodynamic type systems reduce to the form

$$u_t = \left(-\frac{1}{2}\frac{v^3}{w^2}\right)_x, \quad v_t = \left(u + \frac{3}{2}\frac{v^2}{w}\right)_x, \quad w_t = v_x, \tag{21}$$

$$u_y = \left(\frac{1}{4}\frac{v^4}{w^3}\right)_x, \quad v_y = \left(-\frac{1}{2}\frac{v^3}{w^2}\right)_x, \quad w_y = u_x. \tag{22}$$

Their velocity matrices have a single common characteristic root only; such degenerate cases are very interesting. For instance, hydrodynamic type systems with a unique characteristic root were recently investigated [7, 13, 15]. Let us consider the ffirst of the above three-component hydrodynamic type system (21):

$$\begin{aligned} u_t &= -\frac{3v^2}{2w^2}v_x + \frac{v^3}{w^3}w_x, \\ v_t &= u_x + \frac{3v}{w}v_x - \frac{3v^2}{2w^2}w_x, \\ w_t &= v_x. \end{aligned}$$

This system is a first example in the theory of bi-Hamiltonian hydrodynamic type systems where the first Hamiltonian structure has a non degenerate metric tensor, while the second Hamiltonian structure has a degenerate metric tensor. This system possesses an "isometry" extension, i.e.

$$\begin{aligned} u_t &= -\frac{3v^2}{2w^2}v_x + \frac{v^3}{w^3}w_x - x, \\ v_t &= u_x + \frac{3v}{w}v_x - \frac{3v^2}{2w^2}w_x, \\ w_t &= v_x. \end{aligned} \tag{23}$$

Eliminating u and introducing the potential function z such that $w = z_x$ and $v = z_t$, we obtain a single, new third order integrable equation:

$$z_{ttt} = \left(\frac{3z_t^2}{2z_x}\right)_{xt} - \left(\frac{z_t^3}{2z_x^2}\right)_{xx} - 1.$$

The system (23) admits a bi-Hamiltonian pair of the type

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \partial_x, \quad B^{ij} = g^{ij}\partial_x + \Gamma^{ij}_k u^k_x + \epsilon f^i \partial_x^{-1} f^j,$$

which is the $n=3$ analogue of the bi-Hamiltonian pair in 2 components (11). It can be obtained as follows. The metric of the operator B is

$$g^{ij} = \begin{pmatrix} \frac{v^3}{w^2} & \frac{-3v^2}{2w} & -v+1 \\ \frac{-3v^2}{2w} & 2v+1 & w \\ -v+1 & w & 0 \end{pmatrix}, \tag{24}$$

and the isometry that defines the nonlocal part of B is $f = \partial_u$. Explicitely, the operator is:

$$B^{ij} = \begin{pmatrix} \frac{v^3}{w^2} & \frac{-3v^2}{2w} & -v+1 \\ \frac{-3v^2}{2w} & 2v+1 & w \\ -v+1 & w & 0 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} u_x +$$
$$+ \begin{pmatrix} \frac{3v^2}{2w^2} & 0 & 0 \\ -\frac{3v}{w} & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} v_x + \begin{pmatrix} -\frac{v^3}{w^3} & 0 & 0 \\ \frac{3v^2}{2w^2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} w_x + \begin{pmatrix} \partial_x^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The operators A and B are compatible, hence it is possible to write B in Liouville form:

$$B^{ij} = \left(r^{ij} + r^{ji}\right) \partial_x + \frac{\partial r^{ij}}{\partial u^k} u_x^k + f^i \partial_x^{-1} f^j$$

where

$$r^{ij} = \begin{pmatrix} \frac{v^3}{2w^2} & u & 1 \\ -\frac{3v^2}{2w} - u & \frac{1}{2}(2v+1) & 0 \\ -v & w & 0 \end{pmatrix}$$

Moreover, it is possible to write B^{ij} as follows:

$$B^{ij} = \left(\eta^{is} \frac{\partial H^j}{\partial u^s} + \eta^{js} \frac{\partial H^i}{\partial u^s}\right) \partial_x + \eta^{is} \frac{\partial^2 H^j}{\partial u^s \partial u^k} u_x^k + f^i \partial_x^{-1} f^j$$

where

$$H^1(u,v,w) = -uv - \frac{v^3}{2w},$$
$$H^2(u,v,w) = uw + \frac{v^2}{2} + \frac{v}{2},$$
$$H^3(u,v,w) = w.$$

This comes from the fact that the local part B_0 of B is the Lie derivative of A with respect to a vector field, see [9] for details.

One can see, that the second metric tensor (24) can be presented in the following form

$$g^{ij} = \eta^{ij} + \tilde{g}^{ij}, \tag{25}$$

where the metric $\tilde{g}^{ij}$ is degenerate. Thus, we come to the observation that our bi-Hamiltonian system has a compatible pair of Hamiltonian structures, where one of them contains a degenerate metric. This is the first example of this kind in the theory of bi-Hamiltonian systems.

Theorem 6. *The non-homogeneous hydrodynamic type system associated with the bi-Hamiltonian pair A, B can be obtained by appling the operator B to the Casimir u of the operator A. In particular:*

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_t = A^{ij}\frac{\delta F}{\delta u^i} = B^{ij}\frac{\delta L}{\delta u^i}$$

where $L = \int u dx$ *and* $F = \displaystyle\int \left(uv + \frac{v^3}{2w} - \frac{x^2}{2}w \right) dx.$

7 Conclusion

In this paper we considered the new problem of the classification of compatible pairs of Hamiltonian operators $A^{ij} = \eta^{ij}\partial_x$ and $B^{ij} = \tilde{g}^{ij}\partial_x + \tilde{\Gamma}^{ij}_k u^k_x + \epsilon f^i \partial_x^{-1} f^j$.

If the parameter ϵ is zero, then we get back to the classical problem of the description of compatible Hamiltonian operators $A^{ij}_0 = \eta^{ij}\partial_x$ and $B^{ij}_0 = \tilde{g}^{ij}\partial_x + \tilde{\Gamma}^{ij}_k u^k_x$, which was deeply investigated in plenty of papers for past almost forty years (beginning from the seminal article, written by B.A. Dubrovin and S.P. Novikov in 1983, see [3]). We already know a lot of interesting examples of such bi-Hamiltonian pairs. However, the complete description is determined by some integrable systems, which were derived and considered, for instance, in [2], [6], [11]. Even in the 2-component case, such a system is a hydrodynamic type system in 4 dependent variables and 2 independent variables, and a general solution is not known.

In comparison with the above situation, the search of compatible pairs of Hamiltonian operators $A^{ij} = \eta^{ij}\partial_x$ and $B^{ij} = \tilde{g}^{ij}\partial_x + \tilde{\Gamma}^{ij}_k u^k_x + \epsilon f^i \partial_x^{-1} f^j$

when $\epsilon \neq 0$ is a less complicated task. We have been able to completely solve this problem in the two component case, and to give a meaningful example in three components (see Section 6). We are going to continue this investigation in forthcoming publications.

Here we would just like to mention some nontrivial byproducts of our classification. If $\epsilon = 0$, we obtain new examples of compatible Hamiltonian operators $A_0^{ij} = \eta^{ij}\partial_x$ and $B_0^{ij} = \tilde{g}^{ij}\partial_x + \tilde{\Gamma}^{ij}_k u^k_x$. This means that in the general case a compatible pair of *local* Hamiltonian operators of the type of A_0 and B_0 cannot be extended to the case $\epsilon \neq 0$. So, our approach allows to construct new distinguished bi-Hamiltonian structures $A_0^{ij} = \eta^{ij}\partial_x$ and $B_0^{ij} = \tilde{g}^{ij}\partial_x + \tilde{\Gamma}^{ij}_k u^k_x$. Moreover, usually, compatible pairs of these bi-Hamiltonian structures were investigated just if the second metric $\tilde{g}^{ij}$ is non degenerate. However, in our paper we found a list of new examples where such a metric is degenerate, see, for instance, (25).

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