

Computing with Hamiltonian operators

R. Vitolo

Dept. of Mathematics and Physics “E. De Giorgi”
Università del Salento and INFN, Sezione di Lecce
via per Arnesano, 73100 Lecce, Italy
email: raffaele.vitolo@unisalento.it
web: <http://poincare.unisalento.it/vitolo>

Published in Computer Physics Communications, (2019) DOI

Abstract

Hamiltonian operators for partial differential equations are ubiquitous in mathematical models of theoretical and applied physics. In this paper the new Reduce package CDE for computations with Hamiltonian operators is presented. CDE can verify the Hamiltonian properties of skew-adjointness and vanishing Schouten bracket for a differential operator, as well as the compatibility property of two Hamiltonian operators, and it can compute the Lie derivative of a Hamiltonian operator with respect to a vector field. More generally, it can compute with (variational) multivectors, or functions on supermanifolds. This can open the way to applications in other fields of mathematical or theoretical physics.

Keywords: Hamiltonian operators, partial differential equations, integrable systems, Schouten bracket, supermanifolds

Contents

1	Introduction	2
2	Hamiltonian operators and partial differential equations	7
3	Computing with \mathcal{C}-differential operators and superfunctions	11
3.1	CDE and total derivatives	11
3.2	CDE and \mathcal{C} -differential operators	13
4	Five easy pieces	15
4.1	Warm-up: tedious large-scale computations	15
4.2	Finding Darboux coordinates	18
4.3	Compatibility of third-order operators	21
4.4	Finding compatible operators	23
4.5	Computing Lie derivatives	26
5	Conclusions	31

1 Introduction

Hamiltonian operators for partial differential equations (PDEs) are one of the most important tools in the modern theory of integrable systems [1, 24, 68], with applications ranging from pure to applied mathematics and physics. Integrable systems are systems of PDEs for which it is possible to construct classes of exact solutions in closed form. A widely accepted characterization of such systems is that they have infinite sequences of symmetries or conserved quantities in involution, or *hierarchies*.¹ Such quantities play a role in the construction of general solutions (see [1, 24, 68]). It is not an easy task to show that a certain PDE has a hierarchy.

Consider a system of PDEs of the form

$$u_t^i = F^i(x^\lambda, u_\sigma^j), \quad i = 1, \dots, n, \quad (1)$$

where (u^i) are dependent variables, (t, x^λ) are independent variables ($\lambda = 1, \dots, m$),

$$u_\sigma^i = \frac{\partial u^i}{\partial x^{\sigma_1} \dots \partial x^{\sigma_m}}, \quad \sigma \in \mathbb{N}^m \quad (2)$$

(with $u_0^i = u^i$) and F^i is a smooth function of a finite number of arguments. The system (1) is *Hamiltonian* if there exists a linear matrix differential operator (A^{ij}) ,

¹More precisely, they commute with respect to certain bracket operations, see (23)

fulfilling additional properties, such that

$$u_t^i = F^i(x^\lambda, u_\sigma^j) = A^{ij} \frac{\delta H}{\delta u^j}, \quad (3)$$

where H is a conservation law density, the *Hamiltonian density*, for the system (1). The properties that make A a Hamiltonian operator are: A is skew-adjoint, $A^* = -A$, and the Schouten bracket of A with itself vanishes, $[A, A] = 0$. We will come back to those properties in Section 2.

It can be proved [13, 47] that a Hamiltonian operator maps conserved quantities into symmetries. But a Hamiltonian operator alone cannot guarantee the existence of a hierarchy. The fundamental idea of F. Magri [49] was that if a system of PDEs is Hamiltonian with respect to *two* Hamiltonian operators A_1, A_2 , then an infinite sequence of conserved quantities could be generated through the recursive definition

$$A_1^{ij} \frac{\delta H_{k+1}}{\delta u^j} = A_2^{ij} \frac{\delta H_k}{\delta u^j}. \quad (4)$$

The sequence is made of commuting conserved quantities with respect to a certain bracket operation (see (23)) if and only if the operators A_1 and A_2 are *compatible*: $[A_1, A_2] = 0$. Of course, we used the Schouten bracket between operators.

We can pose the main computational problems of the Hamiltonian formalism for PDEs.

1. Given a system of PDEs of the type $u_t^i = F^i(x^\lambda, u_\sigma^j)$, find a Hamiltonian operator and a Hamiltonian density for the system (direct problem).
2. Given a Hamiltonian operator A , find which integrable systems are Hamiltonian with respect to A (inverse problem).
3. Given a differential operator, check the properties that make it Hamiltonian: $A^* = -A$ and $[A, A] = 0$.
4. Given a Hamiltonian operator A , change its coordinates in such a way to achieve a certain canonical form (Darboux theorems for Hamiltonian operators).
5. Given two Hamiltonian operators A_1, A_2 , check their compatibility: $[A_1, A_2] = 0$.
6. Given a Hamiltonian operator A , compute the set of Hamiltonian operators B which are compatible with A : $[A, B] = 0$.
7. Given a pair of compatible Hamiltonian operators A, B , find a vector field τ such that $B = L_\tau A = [\tau, A]$. Here the Lie derivative of A with respect to τ is just the Schouten bracket of the vector field with the operator. This problem is of interest in the theory of Hamiltonian deformations, see subsection 4.5.

While countless papers have been written on the symbolic computation of symmetries and conserved quantities, the literature on symbolic computations with Hamiltonian operators is quite scarce. The same situation holds for publicly available software packages. Possible reasons are that the algorithms for computing with integrability structures, like Hamiltonian, symplectic and recursion operators, are less standard and more involved with respect to those for symmetries and conserved quantities. Another reason might be that few experts in integrable systems have the abilities that are needed to deal with differential operators in a computer algebra system as required for the mathematical problems at hand, and conversely, few experts in computer algebra are also familiar with the mathematics of integrable systems.

A recent book on integrability structures [46] tries to fill the above gaps: it contains, between other material, symbolic computations related to the direct problem (item 1 above). The software package CDE², an official Reduce package [37], is used throughout the book. An extensive review of existing software for integrability structures can be found in the book.

This paper integrates material of [46] in the direction of the solution of the above problems 3–7. It should be remarked that many authors developed software for similar purposes without making it available to the public. That makes their computations much more difficult to reproduce. There are few software packages with similar capabilities that are also in the public domain. We will briefly list them and comment the features that are not already considered in the above mentioned review [46, Section 1.3].

1. The Maple package `jets` [7] contains a module for computing the Schouten bracket of local differential operators. See [46, Section 1.3] for more comments. At the time of writing it seems that the package is no longer available in internet, in particular it is not in the website of the author.
2. The Maple package `Jets` [11], initially developed by Marvan (2003), then also by Baran (2010), has been used for computing Hamiltonian operators for particular differential equations [9, 8, 10, 12]. However, it does not contain any specific feature for integrability structures (like an implementation of the Schouten bracket).
3. The Maple package `JET` [50] can compute with integrability structures, see [46, Section 1.3] for more comments.

The limits of the package are the fact that it can only compute with operators in one independent variable (x) and a partial support of computations with nonlocal (pseudodifferential) operators D_x^{-1} (simplification of such terms cannot always be performed). The recent paper [16] provides an algorithm for the simplification of such expressions; the authors plan to implement the algorithm in computer algebra system (also in CDE) in the near future.

²This is an acronym standing for Calculus on Differential Equations

4. The Maple package `DifferentialGeometry` [2], developed by I. Anderson, is being extended by a library of procedures which are devoted to integrable systems (private communication of I. Anderson to the author). At the moment the library is still not available in Maple.
5. The package [5] deserves a mention, in that it is devoted to finding recursion operators for symmetries; however, the kind of computations that are required are different from those considered here, even if the subject is closely related under the mathematical viewpoint (see [46] for a deeper discussion).
6. In the framework of the algebraic approach to the Hamiltonian formalism for PDEs [6] the Mathematica packages `MasterPVA` and `WAlg` [17] provide procedures for computing the Schouten bracket between local differential operators. The recent paper [16] shows that the algebraic approach to Schouten brackets leads to the same computations and results as more traditional approaches, also for nonlocal operators.

Basically, the only packages that are ready to be used for computations with Hamiltonian operators in a set of problems which is wide enough are the above items 3 and 6. The package `JETS` is limited by the fact that it can only use operators in one independent variable. The package `MasterPVA` uses an algebraic formalism which has a completely different nature to formalisms which are more common in the field. Moreover, both the above packages cannot do the computations of items 4 and 7, because the packages lack of capabilities to linearize an operator and take its adjoint, and cannot deal with multivectors. Indeed, Hamiltonian operators are bivectors, and computing with Hamiltonian operators amounts at doing computations for multivectors in the special case of degree 2, see Section 2.

The above limits result from reading the manuals; maybe with some extra programming both packages might include the above capabilities.

The package `CDE` can compute with operators in an arbitrary number of variables, it uses a standard formalism or the formalism of supermanifolds in order to implement the operators and it can do all computations on Hamiltonian operators that we listed above.

Moreover, `CDE` is able to deal with Hamiltonian operators as variational multivectors. They are dual to variational forms, exactly as ordinary multivectors on finite-dimensional manifolds are dual to differential forms. Variational forms are endowed by the variational differential (that is the analogue of the de Rham differential of forms on manifolds) and constitute the so-called *variational complex* [64]; variational multivectors are endowed by a bracket operation, the Schouten bracket (which is the analogue of the Schouten–Nijenhuis bracket between ordinary multivectors) [39].

Variational multivectors are implemented in `CDE` both as differential operators in several arguments and as superfunctions on supermanifolds. This makes `CDE` unique

within existing software for computing with Hamiltonian operators, also in view of future research with multivectors or in the mathematics and physics of supermanifolds. Indeed, a few packages that can compute with anticommuting (or Grassmann) variables were written so far [18, 36, 61], but none of them has an implementation of all the required capabilities for Hamiltonian operators. At the moment only CDE has an implementation of total derivatives and variational derivatives that act on functions of even or odd field variables and their derivatives. Many calculations that are typical in theoretical physics, like computing the Euler–Lagrange equations of a Lagrangian defined on a supermanifold, computing symmetries and conservation laws for super-differential equations, etc., can be done on a computer by means of CDE, with little or no extra programming. See also [44], where CDIFF, the ‘ancestor’ of CDE, was used with a non-automatic implementation of total derivatives for computing symmetries of super-differential equations; similar tasks were also considered in [4] with a Maple program.

CDE is implemented in Reduce for several reasons: Reduce is free software, and it was possible to study its internals. As a side outcome of this work, many undocumented parts of Reduce internals were described in the manual [55]. Then, Reduce has been used for decades with its package CDIFF for problems which are closely related to those presented in this paper, in particular by Kersten and the University of Twente research group. Finally, CDE has an easy interface that makes the above problems easy to program, with few commands needed to achieve the result.

Nonlocal Hamiltonian operators are an essential part of integrable systems. At the moment, CDE can quite easily compute the conditions for a nonlocal operator to be Hamiltonian for a given partial differential equation (in the sense of mapping conserved quantities into symmetries), see [46] for examples. However, CDE lack of capabilities for computing Schouten brackets of pseudodifferential operators. In order to fill this gap an algorithm was recently developed [16]. A more advanced geometric approach is in development [45]. We hope to extend the range of CDE to nonlocal operators in a near future.

The structure of the paper is as follows.

In Section 2 we describe multivectors and their calculus, also in terms of superfunctions. The isomorphism between multivectors and superfunctions allows to use a very elegant formula for the Schouten bracket.

In Section 3 we describe the CDE implementation of operators and related computations.

In Section 4 we show examples of nontrivial computations performed with CDE. The examples cover the problems 4–7 listed above. While the corresponding results have already been published elsewhere, the computational methods are a fundamental part of the research effort that led to the scientific results, and have never been published before. They can be of help for similar computations in other mathematical problems. Finally, we present several research-grade problems at the end of each

example. Hopefully, they will motivate the Reader to using CDE in his/her scientific research.

2 Hamiltonian operators and partial differential equations

Let us denote independent variables by x^λ , $1 \leq \lambda \leq m$, dependent variables by (u^i) , $1 \leq i \leq n$, and derivatives by u_σ^i (here $\sigma = (\sigma_1, \dots, \sigma_m)$ is a multiindex representing derivatives with respect to x^1 σ_1 times, \dots , x^m σ_m times). For each coordinate x^λ the *total derivative vector field* ∂_λ is defined as

$$\partial_\lambda = \frac{\partial}{\partial x^\lambda} + u_{\sigma, \lambda}^i \frac{\partial}{\partial u_\sigma^i}, \quad (5)$$

(the summation convention holds as usual) where σ, λ stands for the multiindex $\sigma + (0, \dots, 1, \dots, 0)$, where 1 is at the position λ . Note that if $\sigma = 0$ then $u_\sigma^i = u^i$.

Hamiltonian operators are members of a broader family of differential operators, namely, they are *variational multivectors* [13, 34, 57, 39]. Variational multivectors are specific differential operators in total derivatives, or \mathcal{C} -differential operators (see [13] for a definition).

Let us define functional m -forms (or Lagrangians) α as volume forms on the space of independent variables whose coefficients depend on (x^λ, u_σ^i) :

$$\alpha = \alpha_0(x^\lambda, u_\sigma^i) dx^1 \wedge \dots \wedge dx^m \quad (6)$$

The space of functional m -forms is denoted by $\bar{\Lambda}^m$. The kernel of the variational derivative of local functionals consists of *total divergencies*, or functional m -forms of the type $\partial_\lambda(\beta^\lambda) dx^1 \wedge \dots \wedge dx^m$, where (β^λ) , with $1 \leq \lambda \leq m$, is a vector function of (x^μ, u_σ^j) . Let us denote the space of total divergencies by $\bar{d}\bar{\Lambda}^{m-1}$. We define the space of *local functionals* as $\bar{H}^m = \bar{\Lambda}^m / \bar{d}\bar{\Lambda}^{m-1}$.

A variational multivector is defined to be a skew-symmetric \mathcal{C} -differential operator with values in local functionals.

The expression of a variational multivector is

$$\Delta(\psi^1, \dots, \psi^h) = [a^{(\sigma_1 i_1) \dots (\sigma_h i_h)} \partial_{\sigma_1} \psi_{i_1}^1 \dots \partial_{\sigma_h} \psi_{i_h}^h]. \quad (7)$$

Here, the arguments of Δ are vector-valued functions of the type $\psi = (\psi_i(x^\lambda, u_\sigma^j))$, and the coefficients $a^{(\sigma_1 i_1) \dots (\sigma_h i_h)}$ are functions of (x^λ, u_σ^j) which are skew-symmetric with respect to the interchange of pairs $(\sigma_k i_k)$ and $(\sigma_h i_h)$. The value of Δ , i.e., the right-hand side of (7), is an equivalence class up to total divergencies. Note that we will omit $dx^1 \wedge \dots \wedge dx^m$ for the sake of brevity.

A variational multivector can be uniquely represented by taking its formal adjoint in one of its arguments (say the last one; see [13] for a geometric definition):

$$\Delta^*(\psi^1, \dots, \psi^{h-1})^i = (-1)^{|\sigma_h|} \partial_{\sigma_h} (a^{(\sigma_1 i_1) \dots (\sigma_h i)} \partial_{\sigma_1} \psi_{i_1}^1 \dots \partial_{\sigma_{h-1}} \psi^{h-1}) \quad (8)$$

Indeed, it is easy to realize that the above expression is equal to the previous expression (up to a total divergence) and it is divergence-free, i.e., no expression of order zero in one of the arguments can have the form of a total divergence.

The above representation defines an isomorphism between the space of variational h -vectors and the space of vector-valued h - \mathcal{C} -differential operators of the form

$$\square(\psi^1, \dots, \psi^h)^j = b^j(\sigma_1 i_1) \dots (\sigma_h i_h) \partial_{\sigma_1} \psi_{i_1}^1 \dots \partial_{\sigma_h} \psi_{i_h}^h \quad (9)$$

which are both skew-symmetric with respect to the exchange of arguments and skew-adjoint with respect to each argument [39].

The calculus of variational multivectors consists of obvious operations, like sums and compositions, and of the variational Schouten bracket (or just Schouten bracket for short). The Schouten bracket for variational multivectors was first formulated in wide generality in [34] (see also [39, 42]). Its expression is not implemented in CDE in the language of differential operators.

There is another way to express the bracket which is much more elegant and compact [35] (see also [39, 38, 42]). Let us consider a vector of new dependent variables (p_i) , $1 \leq i \leq n$. We will assume that the new variables (and hence their derivatives) are Grassmann, or anticommuting, variables, so that we will be dealing with the jet of a superbundle with coordinates $(x^\lambda, u_\sigma^i, p_{j\tau})$. Then, there is an isomorphism between the space of skew-symmetric vector-valued \mathcal{C} -differential operators (9) and the space of vector-valued superfunctions

$$F^j = b^j(\sigma_1 i_1) \dots (\sigma_h i_h) p_{i_1 \sigma_1} \dots p_{i_h \sigma_h}. \quad (10)$$

In coordinates, the isomorphism is given by

$$b^j(\sigma_1 i_1) \dots (\sigma_h i_h) \partial_{\sigma_1} \psi_{i_1}^1 \dots \partial_{\sigma_h} \psi_{i_h}^h \longrightarrow b^j(\sigma_1 i_1) \dots (\sigma_h i_h) p_{i_1 \sigma_1} \dots p_{i_h \sigma_h} \quad (11)$$

Of course, if the skew-symmetric \mathcal{C} -differential operator is a variational multivector, then the corresponding superfunction is defined up to total divergencies of vector-superfunctions.

Using the above formalism, if two variational multivectors are represented by the scalar superfunctions F and H (up to total divergencies), then we have the following formula for the Schouten bracket:

$$[F, H] = \left[\frac{\delta H}{\delta u^i} \frac{\delta F}{\delta p_i} - (-1)^{(F+1)(H+1)} \frac{\delta F}{\delta u^i} \frac{\delta H}{\delta p_i} \right] \quad (12)$$

Here the letters F , H at the exponent of (-1) mean the Grassmann parity of the corresponding superfunction, and $\delta F/\delta u^i$, $\delta F/\delta p_i, \dots$ denote the variational derivatives

$$\frac{\delta F}{\delta u^i} = (-1)^{|\sigma|} \partial_\sigma \left(\frac{\partial F}{\partial u_\sigma^i} \right), \quad \frac{\delta F}{\delta p_i} = (-1)^{|\sigma|} \partial_\sigma \left(\frac{\partial F}{\partial p_{i\sigma}} \right). \quad (13)$$

Note that the derivatives with respect to odd coordinates are odd derivatives, and total derivatives (5) are extended to odd variables as

$$\partial_\lambda = \frac{\partial}{\partial x^\lambda} + u_{\sigma,\lambda}^i \frac{\partial}{\partial u_\sigma^i} + p_{i\sigma,\lambda} \frac{\partial}{\partial p_{i\sigma}}. \quad (14)$$

The square brackets on the right-hand side of (12) mean that we are in an equivalence class, or that the expression in the bracket is considered up to total divergencies of superfunctions.

The above formula is undoubtedly the clearest and most elegant expression of the Schouten bracket. But its clarity has a price to pay: one should be able to compute with Grassmann variables in order to use it. We will see that CDE implements the required capabilities.

Hamiltonian operators are (matrix) differential operators in total derivatives acting on vector-valued functions $\psi = (\psi_i(x^\lambda, u_\sigma^j))$

$$A(\psi) = A^{ij} \psi_j = a^{i(\sigma j)} \partial_\sigma \psi_j, \quad (15)$$

where $a^{i(\sigma j)}$ are functions of (x^λ, u_σ^k) .

The operator A is requested to fulfill two properties:

- A is formally skew-adjoint: $A^* = -A$, where

$$(A^*)^{ij} \psi_j = (-1)^{|\sigma|} \partial_\sigma (\psi_j a^{j(\sigma i)}) \quad (16)$$

This means that A is a variational bivector represented as (9);

- The *Schouten bracket* of the operator with itself is zero:

$$[A, A] = 0. \quad (17)$$

In order to compute the Schouten bracket of A , we should first of all represent A as a superfunction. In order to use the isomorphism (11) we need to write A in the form (7). This just means that the isomorphism takes the form

$$a^{i(\sigma j)} \partial_\sigma \psi_j^1 \psi_i^2 \longrightarrow a^{i(\sigma j)} p_{j\sigma} p_i, \quad (18)$$

where the product between $p_{j\sigma} p_i$ is the Grassmann product. Then, as the Grassmann parity of A is 2, we have

$$[A, A] = 2 \left[\frac{\delta A}{\delta u^i} \frac{\delta A}{\delta p_i} \right] \quad (19)$$

where $A = a^{i(\sigma j)} p_{j\sigma} p_i$. If we wish to check that the Schouten bracket is zero, we can apply the Euler–Lagrange operator to the expression in the bracket to verify that it is zero.

A partial differential equation in evolutionary form $u_t^i = f^i(x^\lambda, u^j, u_\sigma^j)$ is said to be *Hamiltonian* with respect to a Hamiltonian differential operator $A = (A^{ij})$ if it exists a local functional $H = [h]$, where $h = h(x^\lambda, u_\sigma^i)$, such that

$$u_t^i = f^i(x^\lambda, u^j, u_\sigma^j) = A^{ij} \left(\frac{\delta h}{\delta u^j} \right). \quad (20)$$

It can be proved that a Hamiltonian operator satisfies the following equation:

$$\ell_F \circ A = -A \circ \ell_F^*, \quad (21)$$

where ℓ_F is the Fréchet derivative, or linearization, of the function $F = u_t^i - f^i$ that defines the equation, and ℓ_F^* is its formal adjoint:

$$\ell_F(\varphi) = \frac{\partial F^k}{\partial u_\sigma^i} \partial_\sigma \varphi^i, \quad \ell_F^*(\psi) = (-1)^{|\sigma|} \partial_\sigma \left(\frac{\partial F^k}{\partial u_\sigma^i} \psi_k \right), \quad (22)$$

We observe that the kernel of ℓ_F consists of (generalized, or higher) *symmetries* of the differential equation (20) [13, 57]. A conservation law $\omega = hdt + kdx$ for an evolutionary equation (20) is defined up to trivial quantities of the form $\alpha = \partial_t f dt + \partial_x f dx$, and is uniquely represented by its *generating function*, or *characteristic*, $\psi = (\delta h / \delta u^i)$. The kernel of ℓ_F^* contains generating functions, or characteristic vectors, of conserved quantities [13, 20, 57]. Hence, (21) implies that a Hamiltonian operator maps conserved quantities into symmetries.

An extension of this notion for non-evolutionary equations is also available [43], but will not be considered here.

The *Poisson bracket* of local functionals $H = [h]$ and $F = [f]$ is

$$\{H, F\}_A = \left[\frac{\delta h}{\delta u^i} A^{ij} \frac{\delta f}{\delta u^j} \right]. \quad (23)$$

The Poisson bracket fulfills the properties

$$\{H, F\}_A = -\{F, H\}_A \quad (24)$$

$$\{\{H, F\}_A, G\}_A + \{\{G, H\}_A, F\}_A + \{\{F, G\}_A, H\}_A = 0 \quad (25)$$

and it endows the space of functionals with the structure of a Lie algebra. It can be proved that the above properties are direct consequences of the properties (16) and (17), respectively; and conversely, if the bracket defined by a \mathcal{C} -differential operator A as in (23) fulfills (24) and (25) then A is a Hamiltonian operator. Of course, the bracket can be computed between generating functions of conserved quantities. Usually, a partial

differential equation is said to be *integrable* if it possesses a sequence of conserved quantities which are in involution with respect to a Poisson bracket.

The integrability of a partial differential equation follows when the equation admits two Hamiltonian formulations (20) for two distinct Hamiltonian operators A_1 , A_2 and respective densities H_1 , H_2 ; the operators are required to be *compatible*, or that their Schouten bracket vanish: $[A_1, A_2] = 0$. In this case, the celebrated theorem by Magri [49] states that the conserved quantities generated by the recurrence formula

$$A_1(\psi_{i+1}) = A_2(\psi_i) \tag{26}$$

are in involution with respect to the Poisson brackets defined by both operators: $\{\psi_i, \psi_{i+1}\}_{A_1} = \{\psi_i, \psi_{i+1}\}_{A_2} = 0$, $i = 0, 1, 2, \dots$

3 Computing with \mathcal{C} -differential operators and superfunctions

We describe the implementation of \mathcal{C} -differential operators and superfunctions in CDE and related operations. The exposition differs from that of [46] in that it is aimed at computing with Hamiltonian operators.

Standard Reduce concepts and syntax will not be discussed here; we invite the interested reader to have a look at Reduce’s official website [37]. More details on CDE, including a list of all commands, can be found in [46].

3.1 CDE and total derivatives

Starting from independent, dependent and Grassmann, or odd, variables CDE automatically creates the list of derivative coordinates (as symbols). The basic properties of odd variables in CDE, namely the odd product, the odd derivatives and the definition of a supervectorfield, are provided by the package CDIFF (now officially in the Reduce distribution), written by Gragert, Kersten, Post and Roelofs from the University of Twente (Netherlands), which is automatically loaded by CDE.

In a Reduce terminal, load the CDE package by the command `load_package cde;`. Then a supermanifold is created by the command

```
cde({indep_var, dep_var, odd_var, total_order}, {});
```

Here, the first list contains the names of the lists of independent variables, dependent variables, odd variables and the maximal order of derivative coordinates. For example, we might have

```
indep_var:={t,x};  dep_var:={u,v};
odd_var:={p,q};  total_order:=10;
```

The empty list at the second argument of `cde` is used for the restriction of total derivatives on differential equations (which we will not use in this paper). After the above command the system has distinct lists of even derivatives (u_σ^i) and odd derivatives ($p_{j\tau}$) (up to the order `total_order`) in the form

```
u_x,v_t,u_t2x,v_tx,v_2t3x,...,p_t2x,...
```

Note that the symbol `v_xt` does not exist in CDE (with the above data) for performance reasons. Odd variables can appear in anticommuting products; this is represented as

```
ext(p,p_2tx),ext(p_x,q_t,q_t2x),...
```

where `p` and `ext(p)` are just the same, while `ext(p_2tx,p)` is not automatically expanded to `-ext(p,p_2tx)` for performance reasons; only one of the two symbols is understood by the system. In order to get the correct representation the function

```
odd_product(phi,psi);
```

should be used for the odd product of two superfunctions. The derivative of an expression `phi` with respect to an odd variable `p` is achieved by `df_odd(phi,p);`.

CDE also defines total derivatives (14) on the given supermanifold, truncated at the order `total_order`. Total derivatives are distinguished supervector fields; they are defined through a CDIFF function using data that is constructed by CDE. Basically, for each derivative symbol (like `u_2tx`) in the system the coefficient of the corresponding derivative $\partial/\partial u_{2tx}$ is computed by increasing the multiindex by one in the direction of the corresponding independent variable, and the results are summed up to make (14). The total derivative of a superfunction `phi` is invoked by `td(phi,x);` or `td(phi,t,x,2);`, for example. The syntax closely follows Reduce syntax for standard derivatives `df`; the above expressions translate to $\partial_x\varphi$ and $\partial_t\partial_x^2\varphi$, respectively.

Each time that a coefficient in a total derivative has order higher than `total_order` it is replaced by the identifier `letop`³, which is a function that depends on all independent variables. After evaluating total derivatives, the result is scanned for the presence of `letop`. If that is the case the program stops with an error message and the computation must be repeated with a higher value of `total_order`. If needed, CDE programs can be run by a Linux shell script, `rrr.sh`, which is included in the package; the script will re-run the program with a higher jet order if a jet order error is met.

The computation of total derivatives can lead to huge expressions. It is possible to disable the expansion of total derivatives by the command `noexpand_td();` (and re-enable it by `expand_td();`).

The command

```
pvar_df(par,expr,dvar)
```

³In Dutch ‘let op’ means ‘pay attention’

computes the variational derivative $\delta F/\delta w^i$, where F is `expr`, w^i is the dependent variable `dvar` and `par` is the parity of the dependent variable, 0 if even or 1 if odd. The command `euler_df(expr)` computes all even and odd variational derivatives and returns their values in a list of two lists.

3.2 CDE and \mathcal{C} -differential operators

A vector valued \mathcal{C} -differential operator (9) must be declared in CDE as follows:

```
mk_cdifop(opname,num_arg,length_arg,length_target);
```

where

- `opname` is the name of the operator (\square in (9));
- `num_arg` is the number of arguments (h in (9));
- `length_arg` is the list of lengths of the arguments, e.g., in (9) one needs a list of h items $\{\mathbf{k}_1, \dots, \mathbf{k}_h\}$, each corresponding to number of components of the vector functions $\psi_{i_j}^j$ to which the operator is applied. In the calculations of this paper we will only use one argument;
- `length_target` is the number of components of the image vector function (the range of the index j in (9)).

The above parameters of the operator `opname` are saved in the *property list* of the identifier `opname` (more comments can be found in [46]). The value of one component of the operator \square on the arguments ψ^1, \dots, ψ^h is

```
opname(j,i1,...,ih,psi1,...,psih);
```

A vector-valued superfunction (10) must be declared in CDE as follows:

```
mk_superfun(sfname,deg,length_target);
```

where

- `sfname` is the name of the superfunction (F in (10));
- `deg` is the degree of the superfunction, e.g., h in (10);
- `length_target` is the number of components of the image vector (the range of the index j in (10)).

The syntax for one component of the superfunction `sfname` is

```
sfname(j);
```

CDE can convert \mathcal{C} -differential operators with one argument into superfunctions of degree 1 and back:

- `conv_cdifftosuperfun(cdop,superfun)`
- `conv_superfun2cdifftop(superfun,cdop)`

The translation from an operator to a superfunction is easy: it is enough to evaluate all components of the operator on all odd dependent variables. The other direction is more complicated: coefficients $C_i^{j\sigma}$ of odd variables $p_{j\sigma}$ have to be collected from each component i of the superfunction, then a Reduce operator `cdop(i,j,psi)` is defined by the sum (with respect to σ) $C_i^{j\sigma} \partial_\sigma \psi$. Here the technical difficulty is defining an operator inside a Reduce procedure, `psi` being just a formal parameter and not an expression. Indeed, as a data structure, a superfunction is much more easy to handle than a Reduce operator.

The linearization of a purely even vector-function and the adjoint of a \mathcal{C} -differential operator with one argument are computed by means of the isomorphism (11). More precisely, given a function $F = F(x^\lambda, u_\sigma^i)$ the following superfunctions can be easily computed:

$$\ell_F((p_j))^k = \sum_{i,\sigma} \frac{\partial F^k}{\partial u_\sigma^i} p_{i\sigma} \quad \ell_F^*((p_j))_i = (-1)^{|\sigma|} \partial_\sigma \left(\frac{\partial F^k}{\partial u_\sigma^i} p_{k\sigma} \right) \quad (27)$$

and then translated back into \mathcal{C} -differential operators using the above conversion utilities.

In CDE a vector function must be introduced as a list of scalar functions

```
fun:={fun1,fun2,...};
```

Then its linearization is achieved by the left formula (27) implemented in the command

```
ell_function(fun,lfun);
```

where `lfun` is automatically declared as a \mathcal{C} -differential operator with the appropriate parameters. Moreover, the above command creates a superfunction `lfun_sf` that corresponds to the \mathcal{C} -differential operator `lfun`.

The command

```
adjoint_cdifftop(lfun,lfun_star);
```

computes the adjoint `lfun_star` of `lfun` by the formula on the right of (27) and introduces an equivalent superfunction whose identifier has the suffix `_sf`: `lfun_star_sf`.

4 Five easy pieces

Elementary computations with Hamiltonian operators can be found in the CDE section in Reduce’s manual [37]. Here, we would like to show the solution of the computational problems that we listed in the Introduction (p. 3, items 4–7) for nontrivial examples. The examples have been published in separate research papers in recent years.

While results are almost all known, a detailed description of the corresponding computations appears here for the first time. The design of effective software for the solution of the mathematical problems that have been considered so far was a fundamental and nontrivial part of the research activity. It deserves a separate exposition and it might be of interest when trying to solve similar problems. All the examples are provided as Reduce program files in the website of the author <http://poincare.unisalento.it/vitolo>.

To the authors’ knowledge, no other computer algebra computations of this type and level have been published with all details before.

At the end of each example a research problem is presented in order to motivate the Reader to use CDE in his/her research.

4.1 Warm-up: tedious large-scale computations

The associativity equation, or Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equation was derived in the context of $2D$ -topological field theory. It consists in an overdetermined system of PDEs with one dependent variable and N independent variables. Nowadays its significance is mostly mathematical. For example, its solutions yield, under mild hypotheses, bi-Hamiltonian systems. See [26] for a mathematical introduction to the equation.

In [29] a bi-Hamiltonian formulation of the WDVV equation in the simplest case $N = 3$ was found. It was only in recent times that such a result was extended to the case $N = 4$, also thanks to CDE [59]. Here we will describe the role of CDE in [59].

In [30] the WDVV equation was rewritten, in the case $N = 4$ (see [26] for details), as a pair of commuting hydrodynamic-type systems

$$\begin{array}{ll}
 a_y^1 = a_x^2, & a_z^1 = a_x^3 \\
 a_y^2 = a_x^4, & a_z^2 = a_x^5 \\
 a_y^3 = a_x^5, & a_z^3 = a_x^6 \\
 a_y^4 = R_x, & a_z^4 = P_x, \\
 a_y^5 = P_x, & a_z^5 = S_x, \\
 a_y^6 = S_x, & a_z^6 = Q_x
 \end{array} \quad \text{and} \quad (28)$$

where

$$P = \frac{a^3 a^4 + a^6}{a^1}, \quad R = \frac{2a^5 + a^2 a^4}{a^1}, \quad S = \frac{2a^3 a^5 - a^2 a^6}{a^1},$$

$$Q = (a^5)^2 - a^4 a^6 + \frac{(a^3)^2 a^4 + a^3 a^6 - 2a^2 a^3 a^5 + (a^2)^2 a^6}{a^1}.$$

In [30] (see also [54]) it was proved that the above two systems (28) admit a first-order Dubrovin–Novikov Hamiltonian operator. This is an operator of the type

$$A_1^{ij} = h^{ij} \partial_x + \Gamma_k^{ij} u_x^k. \quad (29)$$

The operator is homogeneous (of degree 1) with respect to a grading which is given by x -derivatives [27]. This implies that the form of the operator is invariant with respect to coordinate transformations of type $\tilde{u}^i = \tilde{u}^i(u^j)$. As a consequence, the coefficients transform as geometric objects: for example, h^{ij} transforms as a contravariant 2-tensor. It can be proved that the Hamiltonian property of the operator is equivalent (here and in what follows $\det(h^{ij}) \neq 0$) to the fact that $(h_{ij}) = (h^{ij})^{-1}$ is a flat pseudo-Riemannian metric and $\Gamma_{hk}^j = -h_{hi} \Gamma_k^{ij}$ are its Christoffel symbols.

In our example, the first-order Dubrovin–Novikov Hamiltonian operator for (28) is

$$A_1^{ij} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ a^1 & a^2 & a^3 & a^4 & a^5 & a^6 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ a^2 & a^4 & a^5 & R & P & S \\ 2a^3 & 2a^5 & 2a^6 & 2P & 2S & 2Q \end{pmatrix} \partial_x + \partial_x \begin{pmatrix} 0 & 0 & a^1 & -1 & a^2 & 2a^3 \\ 0 & -1 & a^2 & 0 & a^4 & 2a^5 \\ 0 & 0 & a^3 & 0 & a^5 & 2a^6 \\ -1 & 0 & a^4 & 0 & R & 2P \\ 0 & 0 & a^5 & 0 & P & 2S \\ 0 & 0 & a^6 & 0 & S & 2Q \end{pmatrix}. \quad (30)$$

In [59] it was proved that the above two systems (28) are indeed bi-Hamiltonian, as they also admit the compatible Dubrovin–Novikov type third-order Hamiltonian operator [25] (see subsection 4.2)

$$A_2^{ij} = \partial_x (g^{ij} \partial_x + c_k^{ij} a_x^k) \partial_x, \quad (31)$$

where we assume that $\det(g^{ij}) \neq 0$, and if we set $(g_{ij}) = (g^{ij})^{-1}$, then

$$g_{ik}(\mathbf{a}) = \begin{pmatrix} (a^4)^2 & -2a^5 & 2a^4 & -(a^1 a^4 + a^3) & a^2 & 1 \\ -2a^5 & -2a^3 & a^2 & 0 & a^1 & 0 \\ 2a^4 & a^2 & 2 & -a^1 & 0 & 0 \\ -(a^1 a^4 + a^3) & 0 & -a^1 & (a^1)^2 & 0 & 0 \\ a^2 & a^1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (32)$$

The coefficients c_k^{ij} are given by the formula [31]

$$c_{skm} = \frac{1}{3} (g_{sm,k} - g_{sk,m}), \quad (33)$$

where $c_{ijk} = g_{iq} g_{jp} c_k^{pq}$. The condition $[A_2, A_2] = 0$ is ensured by the properties [31, 62]

$$g_{mk,s} + g_{ks,m} + g_{ms,k} = 0, \quad (34)$$

$$c_{msk,l} = -g^{pq}c_{pml}c_{qsk}. \quad (35)$$

In [59] CDE was used to check that $[A_1, A_2] = 0$. Indeed, for few types of Dubrovin–Novikov operators (like first, second and third-order operators) it is known that the vanishing of the Schouten bracket is equivalent to some tensorial conditions on the coefficients of the operators. However, at the moment of writing, tensorial conditions of compatibility of a first order and a third-order Dubrovin–Novikov operator are not known. Hence, the only way to check that $[A_1, A_2] = 0$ is a direct computation. This is a very long and tedious task which is not instructive at all: a typical computation for a machine.

Here we describe the implementation of the calculation of $[A_1, A_2] = 0$. The program file is `w6c_biham` and can be found at the web page of the author <http://poincare.unisalento.it/vitolo>.

In Reduce, load the package CDE, then declare the input variables and call `cde`:

```
indep_var:={x};
dep_var:={a,b,c,d,ee,f};
odd_var:={p,q,r,s,tt,u};
total_order:=10;
cde({indep_var,dep_var,odd_var,total_order},{});
```

Then we define two matrices whose entries are: the metric of the first-order operator (in upper indices) `hu1(i,j)`, and the metric of the third-order operator (in lower indices) `gl3(i,j)`. We define two operators, `gamma_hi_con(i,j)` that contains the expression $\Gamma_k^{ij}u_x^k$ and `c_hi_con(i,j)` that contains the expression $c_k^{ij}u_x^k$. The operator A_1 is defined as

```
mk_cdiffof(aa1,1,{6},6);
for all i,j,psi let aa1(i,j,psi)=
  hu1(i,j)*td(psi,x)+gamma_hi_con(i,j)*psi;
```

and the operator A_2 is defined as:

```
mk_cdiffof(aa2,1,{6},6);
for all i,j,psi let aa2(i,j,psi) =
  td(
  gu3(i,j)*td(psi,x,2)+c_hi_con(i,j)*td(psi,x)
  ,x);
```

We convert them into superfunctions, according with (11)

```
conv_cdiffof2superfun(aa1,aa1_sf);
conv_cdiffof2superfun(aa2,aa2_sf);
```

We take their adjoint by `cde` and make the following simple skew-adjointness test:

```

adjoint_cdiffof(aa1,aa1_star);
for i:=1:length(dep_var) do
  if aa1_sf(i) + aa1_star_sf(i) neq 0 then
    write "Warning: non-skew-adjoint operator!";

```

and analogously for aa2. Then, we convert the operators into bivectors, according with (9)

```

conv_genfun2biv(aa1_sf,biv1);
conv_genfun2biv(aa2_sf,biv2);

```

Finally, we should check whether the Schouten bracket of the two operators is zero. We can even compute all possible Schouten brackets, to check the Hamiltonian property of the two operators:

```

iszero_schouten_bracket(biv1,biv1,thr11b);
iszero_schouten_bracket(biv1,biv2,thr12b);
iszero_schouten_bracket(biv2,biv2,thr22b);

```

the results are lists of zeros, and the computation takes a negligible time on a contemporary laptop of average computing power.

1 Problem. When forming WDVV equations an essential parameter is the number N of independent variables. It is an opinion of the author that WDVV equations might have a bi-Hamiltonian formalism for an arbitrary value of N . It is a recent discovery that the same holds for another system of PDEs that is of fundamental importance in the geometric theory of integrable systems, the oriented associativity equation, in the case $N = 3$ [15].

The Readers might wish to try to find a bi-Hamiltonian formalism for the WDVV system in the easiest unknown case $N = 5$, to corroborate (or negate!) the above conjecture. The recommended literature for this problem is [29, 33, 59].

4.2 Finding Darboux coordinates

A natural problem of the theory of Hamiltonian operators is: *given a Hamiltonian operator A , find coordinates such that the operator takes the form $A^{ij} = \eta^{ij} \partial_x$, where (η^{ij}) is a constant matrix.* The corresponding coordinates are said to be *Darboux coordinates* of A . The problem of finding Darboux coordinates for Hamiltonian operators was considered by many authors so far, like [3, 53, 52, 56], where scalar Hamiltonian operators have been considered.

Darboux coordinates for operators A_1 of the type (29) always exist: they are flat coordinates for the flat pseudo-Riemannian metric h_{ij} . This means that a point transformation (which is not always easy to find in concrete examples) is enough to transform an operator of type (29) to $A^{ij} = \eta^{ij} \partial_x$.

A more difficult problem is finding Darboux coordinates for higher-order Dubrovin–Novikov operators. These operators were introduced in [25]. In the third-order case they have the form

$$A^{ij} = g^{ij} \partial_x^3 + b_k^{ij} u_x^k \partial_x^2 + (c_k^{ij} u_{xx}^k + c_{km}^{ij} u_x^k u_x^m) \partial_x + d_k^{ij} u_{xxx}^k + d_{km}^{ij} u_{xx}^k u_x^m + d_{kmn}^{ij} u_x^k u_x^m u_x^n. \quad (36)$$

where coefficients are functions of (u^i) . It can be proved that the coefficient $-g_{js} d_k^{is}$ transforms like a linear connection that is symmetric and flat by the Hamiltonian property. The operator (36) can be rewritten in flat coordinates of $-g_{js} d_k^{is}$ as in (31). This means that the operator is completely determined by its leading term (g^{ij}) using (33). The pseudo-Riemannian metric (g_{ij}) is in bijective correspondence with certain projective varieties, see [31, 32, 33]. In particular, in the case of 3 dependent variables u^1, u^2, u^3 the operators are divided in 6 classes with respect to reciprocal transformations of the following projective type

$$d\tilde{x} = (a_0^i + a_i^0 u^i) dx, \quad d\tilde{t} = dt, \quad \tilde{u}^i = \frac{a_0^i + a_j^i u^j}{a_0^0 + a_j^0 u^j}, \quad (37)$$

where $a_j^i, a_i^0, a_0^i, a_0^0$ are constants. The 6 projective classes of operators are

$$\begin{aligned} g^{(1)} &= \begin{pmatrix} (u^2)^2 + c & -u^1 u^2 - u^3 & 2u^2 \\ -u^1 u^2 - u^3 & (u^1)^2 + c(u^3)^2 & -cu^2 u^3 - u^1 \\ 2u^2 & -cu^2 u^3 - u^1 & c(u^2)^2 + 1 \end{pmatrix}, \\ g^{(2)} &= \begin{pmatrix} (u^2)^2 + 1 & -u^1 u^2 - u^3 & 2u^2 \\ -u^1 u^2 - u^3 & (u^1)^2 & -u^1 \\ 2u^2 & -u^1 & 1 \end{pmatrix}, \quad g^{(3)} = \begin{pmatrix} (u^2)^2 + 1 & -u^1 u^2 & 0 \\ -u^1 u^2 & (u^1)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ g^{(4)} &= \begin{pmatrix} -2u^2 & u^1 & 0 \\ u^1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g^{(5)} = \begin{pmatrix} -2u^2 & u^1 & 1 \\ u^1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad g^{(6)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

After introducing potential coordinates $u^i = b_x^i$ the operator (31) takes the form

$$A^{ij} = - (g^{ij} \partial_x + c_k^{ij} b_{xx}^k), \quad (38)$$

where coefficients are functions of (b_x^i) . It can be proved [59, 33] that each of the above metrics can be factorized as $g_{ij} = \varphi_{\alpha\beta} \psi_i^\alpha \psi_j^\beta$, where $(\varphi_{\alpha\beta})$ is a constant non-degenerate symmetric matrix and ψ_i^α are linear functions of the field variables: $\psi_i^\alpha = \psi_{im}^\alpha u^m + \omega_i^\alpha$, where $\psi_{im}^\alpha = -\psi_{mi}^\alpha$. In [33] we proved that n Casimirs exist for every operator in the form (38). More precisely, we proved that the functions $C^\alpha = (\frac{1}{2} \psi_{mk}^\alpha b_x^k + \omega_m^\alpha) b^m$ satisfy

$$A^{ij} \frac{\delta C^\alpha}{\delta b^j} = 0. \quad (39)$$

The above Casimirs turn out to be Darboux coordinates for $g^{(4)}$ and $g^{(5)}$. In order to prove that, we use the change of coordinates formula (see, e.g., [53, 56])

$$\tilde{A} = \ell_C \circ A \circ \ell_C^*, \quad (40)$$

where C is the Casimir vector function. The proof that \tilde{A} will be of Darboux type can be done by CDE. We refer to the program file `casimir`. We start by

```
indep_var:={x};
dep_var:={b1,b2,b3};
odd_var:={p1,p2,p3};
total_order:=6;
```

and call `cde`, then we load the metric

```
g_5:=mat((- 2*b2_x,b1_x,1),(b1_x,1,0),(1,0,0));
gl3:=g_5;
gu3:=gl3**(-1);
```

and define the operator `c_hi_con(i,j)` with the values of the expression $c_k^{ij} b_{xx}^k$ (see the source file) and introduce the operator A as

```
mk_cdifop(a,1,{3},3);
for all i,j,psi let a(i,j,psi) =
- (gu3(i,j)*td(psi,x)+c_hi_con(i,j)*psi);
```

The Casimirs of the operator A are:

```
operator casimir;
casimir(1):=b1;
casimir(2):=b2;
casimir(3):=b3 + b1_x*b2;
```

We linearize the vector function `casimir` and define its adjoint:

```
f_dar:=for i:=1:ncomp collect casimir(i);
ell_function(f_dar,ldar);
adjoint_cdifop(ldar,ldar_star);
```

The formula (40) is easily implemented:

```
mk_cdifop(ta,1,{3},3);
for all i,j,psi let ta(i,j,psi)=
for k:=1:ncomp sum for h:=1:ncomp sum
ldar(i,k,a(k,h,ldar_star(h,j,psi)));
```

The result can be converted into a superfunction for a better presentation

```
conv_cdifff2superfun(ta,ta_sf);
```

we have

```
ta_sf(1);
- p3_x
ta_sf(2);
- p2_x
ta_sf(3);
- p1_x
```

that confirms that the above choice of Casimirs is a set of Darboux coordinates for A . In particular,

$$\tilde{A} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \partial_x. \quad (41)$$

Note that this is a proper differential substitution, i.e., a transformation that depends on derivatives of the dependent variables.

2 Problem. Proving that the nonlocal Casimirs are Darboux coordinates for $g^{(4)}$ is not a problem [41, 40]. However, the author did not manage to prove a similar result for the metrics $g^{(1)}$, $g^{(2)}$, $g^{(3)}$ and the metric of the third-order operator in subsection 4.1. Note that the equation (40), where $\tilde{A} = \eta^{ij} \partial_x$, η^{ij} is a constant matrix and C is an unknown vector function, is non-linear with respect to C ; the resulting system of equations on C can be difficult to solve, even by using a simplified ansatz. The interested Reader might wish to consider this as a possible research problem.

4.3 Compatibility of third-order operators

A compatible pair A_1, A_2 of Hamiltonian operators immediately leads to new integrable systems. The standard way to construct them is to solve the equation for Casimirs of A_1 :

$$A_1^{ij} \frac{\delta C}{\delta u^j} = 0, \quad (42)$$

where C is an unknown function of (u^k) . If there are at least n independent solutions of the above equations, they can be used as new coordinates (\tilde{u}^i) . In most cases the new coordinates are *not* Casimirs of A_2 , and define an integrable hierarchy through Magri's recursion (26).

For this reason, an interesting question that can be posed is: if we consider all possible pairs of homogeneous third-order Hamiltonian operators A_1, A_2 , both from our list on page 19 (in the form (31)), will there be compatible pairs? The answer is: yes, but only in a rather trivial sense. The details are in the following table:

	$g^{(1)}$	$g^{(2)}$	$g^{(3)}$	$g^{(4)}$	$g^{(5)}$	$g^{(6)}$
$g^{(1)}$	y^*	n	n	n	n	n
$g^{(2)}$	n	y	n	n	n	n
$g^{(3)}$	n	n	y	y	n	n
$g^{(4)}$	n	n	y	y	n	n
$g^{(5)}$	n	n	n	n	y	n
$g^{(6)}$	n	n	n	n	n	y

The notation is: n for non compatible, y for compatible, y^* for compatible under additional conditions. More particularly, the operators coming from $g^{(3)}$ and $g^{(4)}$ are indeed the direct sum of a 2×2 block and the one-dimensional operator ∂_x^3 , and the 2×2 blocks are known to be compatible [31]. Operators A_1 and A_2 coming from $g^{(1)}$ are compatible if and only if the value of the constants c in both operators is the same; so, we have no new compatible pairs besides the known ones.

The code for the above computation is rather simple (file `compat3rd`). We define an operator by

```
mk_cdifop(aa1,1,{ncomp},ncomp);
for all i,j,psi let aa1(i,j,psi) =
td(
  gu3_1(i,j)*td(psi,x,2) + c_hi_con_1(i,j)*td(psi,x)
  ,x
);
```

and another operator `aa2` with a different metric `gu3_2(i,j)`, and then convert both to superfunctions and bivectors:

```
conv_cdif2superfun(aa1,aa1_sf);
conv_cdif2superfun(aa2,aa2_sf);
conv_genfun2biv(aa1_sf,biv_aa1);
conv_genfun2biv(aa2_sf,biv_aa2);
```

Finally, we compute the Schouten bracket and require that the three-vector is a total divergence:

```
schouten_bracket(biv_aa1,biv_aa2,th12);
eth12:=euler_df(th12(1));
templ:=for each el in eth12 join el;
sb_coeff:=splitext_list(templ);
sb_num_coeff:=for each el in sb_coeff collect num el;
sb_allcoeff:=splitvars_list(sb_num_coeff,all_parametric_der);
```

The list `eth12` contains all variational derivatives with respect to even and odd coordinates of the Schouten bracket, the list `sb_coeff` contains all coefficients of odd variable

expressions, the list `sb_num_coeff` contains the numerators of the previous expressions, and the list `sb_allcoeff` contains the coefficients of monomials of even variables. It is easy to deduce the above results.

Despite the fact that the result is negative, it is new and the above computational scheme can be used for more interesting computations. The above computation would clearly be very hard with pen and paper.

3 Problem. The above computation is not a well-posed problem under the geometric viewpoint. Indeed, if A and B are two third-order Hamiltonian operators of the type (31) then only one of them (say A) can be brought to one of the forms (4.2); the other will be again of the form (31) but the leading term will not necessarily be in the list (4.2).

It is an open problem to check if there are any compatible pairs A, B , where A is one of the operators in the above list and B is another operator from the same list after a transformation of type (37).

4.4 Finding compatible operators

In this subsection we will address the following problem: given a Hamiltonian operator A , find all Hamiltonian operators B that are compatible with A . As it is known (see also the beginning of the previous subsection), compatible pairs of Hamiltonian operators yield integrable systems [49]. Usually, the answer is provided for specified forms of the unknown operator B .

We consider the above problem in the following formulation: given a third-order local homogeneous Hamiltonian operator R find all first-order homogeneous Hamiltonian operators P that are compatible with R : $[P, R] = 0$. This problem was completely solved in the case $n = 2$ in [48]. In this case there is an affine classification of third-order homogeneous Hamiltonian operators [31]: they are the three operators R_1, R_2, R_3 respectively determined by the three pseudo-Riemannian metrics

$$g^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad g^{(2)} = \begin{pmatrix} -2u^2 & u^1 \\ u^1 & 0 \end{pmatrix}, \quad g^{(3)} = \begin{pmatrix} (u^2)^2 + 1 & -u^1 u^2 \\ -u^1 u^2 & (u^1)^2 \end{pmatrix}.$$

One of the results from [48] is that P_1 is a Hamiltonian operator compatible with R_3 if and only if

$$g_1^{11} = c_1 u^1 + c_2 u^2 + c_3, \tag{43a}$$

$$g_1^{12} = c_4 u^1 - \frac{c_2}{2u^1} + \frac{c_3 u^2}{u^1} + \frac{c_2 (u^2)^2}{2u^1}, \tag{43b}$$

$$g_1^{22} = 2c_4 u^2 + \frac{c_1}{u^1} + \frac{c_5 u^2}{u^1} - \frac{c_1 (u^2)^2}{u^1} + c_6, \tag{43c}$$

where g_1^{ij} are the coefficients of the metric of the first-order operator P_1 (of the form (29)) together with the algebraic conditions

$$c_2c_5 + 2c_1c_3 = 0, \quad c_2c_6 - 2c_3c_4 = 0, \quad c_1c_6 + c_4c_5 = 0. \quad (44)$$

Let us set up the computation in CDE; we describe the program file `compat13`. After initialization, we define an operator `hu1_op` that contains the leading term of the unknown operator P_1 : `hu1_op(i,j):=hu1_ij` where `hu1_ij` depends on dependent variables only. We also define an operator `gamma_hi` in such a way that

1. `gamma_hi(i,j,k):=gamma_hi_ijk` for $i < j$
2. `gamma_hi(i,i,k):=(1/2)*df(hu1_op(i,i),part(dep_var,k))`
3. `gamma_hi(j,i,k):= - gamma_hi(i,j,k) + df(hu1_op(i,j),part(dep_var,k))`
for $i > j$

Indeed, we can use the linear part of the Hamiltonian properties for an operator of the form (29) as given in [21] in order to reduce the number of unknowns: they are

$$\Gamma_k^{ij} + \Gamma_k^{ji} = \partial_k g_1^{ij} \quad (45)$$

Then, we introduce an operator `gamma_hi_con` such that `gamma_hi_con(i,j)` contains the expression $\Gamma_k^{ij} u_x^k$. Now, we define the first-order operator

```
mk_cdifop(aa1,1,{2},2);
for all i,j,psi let aa1(i,j,psi)=
  hu1_op(i,j)*td(psi,x)+gamma_hi_con(i,j)*psi;
```

(here `aa1` stands for P_1) and from the metric

```
matrix g2_3(2,2);
g2_3(1,1):=b2**2 + 1;
g2_3(1,2):= - b1*b2;
g2_3(2,1):=g2_3(1,2);
g2_3(2,2):=b1**2;
g13:=g2_3;
```

construct the third-order operator

```
mk_cdifop(aa2,1,{2},2);
for all i,j,psi let aa2(i,j,psi) =
  td(
  gu3(i,j)*td(psi,x,2)+c_hi_con(i,j)*td(psi,x)
  ,x);
```


(here aa2 stands for R_3). After converting the operators into bivectors

```
conv_cdifff2superfun(aa1,sym1);
conv_cdifff2superfun(aa2,sym2);
conv_genfun2biv(sym1,biv1);
conv_genfun2biv(sym2,biv2);
```

it can be easily checked that $[R_3, R_3] = 0$:

```
schouten_bracket(biv2,biv2,sb22);
euler_df(sb22(1));
```

The compatibility equation $[P_1, R_3] = 0$ is found by

```
schouten_bracket(biv1,biv2,sb12);
comp12:=euler_df(sb12(1));
```

Then, we solve the above equation. To do that, we first define an operator `equ` whose values are the 4 components of the equation, then use a CDE procedure that takes all coefficients of all monomials of odd coordinates in the equations and put them in the values `equ(5)`, `equ(6)`, ... of the operator:

```
operator equ;
equ(1):=num first first comp12;
equ(2):=num second first comp12;
equ(3):=num first second comp12;
equ(4):=num second second comp12;
tel:=4;
tel_start:=4;
tel:=splitext_opequ(equ,1,4);
```

then, we set up the solver CRACK for overdetermined PDEs [66, 65, 67]

```
unk:=append(unk_hu1,unk_gamma_hi);
system_eq:=for i:=tel_start+1:tel collect equ(i);
load_package crack;
lisp(max_gc_counter:=10000000000);
crack_results:=crack(system_eq,{},unk,
  cde_difflist(all_parametric_der,dep_var));
```

the solution is (43) (after renaming the constants); we are also able to determine the Christoffel symbols Γ_k^{ij} in terms of the same constants. We should require the further condition

$$g^{is}\Gamma_s^{jk} = g^{js}\Gamma_s^{ik} \quad (46)$$

in order to obtain the symmetry of Γ_{jk}^i and the fact that Γ_{jk}^i are the Christoffel symbols of the Levi-Civita connection of g_1^{ij} . It turns out that the metric in (43) and the symbols Γ_k^{ij} satisfy the above equation without further conditions. Moreover, we must require the flatness of the metric g_1^{ij} ; that amounts to the algebraic equations (44).

In [48] we also prove that there are pencils of compatible first-order operators inside the space of solutions of the above problem. Unpublished computations with CDE show that such pencils continue to exist in the case $n = 3$ and $n = 4$, thus providing multi-parameter spaces of Miura-inequivalent integrable systems according with the mechanism which was first discovered in [58]. Such results would be very hard to achieve without computer-assisted calculations.

4 Problem. Find the list of all pencils of compatible first-order operators that are compatible with a fixed third-order operator in the case $n = 3$ or $n = 4$. Use the classification of third-order operators provided in [31, 32].

4.5 Computing Lie derivatives

Given a pair of compatible Hamiltonian operators A_1, A_2 , it happens in many cases that there exists a vector field

$$\tau = \tau^i \frac{\partial}{\partial u^i}, \quad \text{where} \quad \tau^i = \tau^i(x^\lambda, u_\sigma^i), \quad (47)$$

such that $A_2 = L_\tau A_1 = [\tau, A_1]$, where the bracket is obviously the Schouten bracket. This is more than just a curiosity: it is an essential feature of the perturbative approach to the classification of integrable systems in 1 + 1 dimensions initiated in [22]. Indeed, consider a Hamiltonian deformation of a first-order homogeneous Hamiltonian operator P_0 of the type (29)

$$P_\epsilon = P_0 + \epsilon P_1 + \epsilon^2 P_2 + \dots \quad (48)$$

i.e., a one-parameter family of Hamiltonian operators P_ϵ where each of the summands P_1, P_2, \dots is homogeneous of increasing degree. Then, it can be proved under some reasonable assumptions (see [19, 51]) that the deformation is always trivial, in the sense that there exists a vector field τ such that $P_\epsilon = L_\tau P_0$; this implies the existence of a formal diffeomorphism φ_ϵ such that $P_\epsilon = \ell_{\varphi_\epsilon} \circ P_0 \circ \ell_{\varphi_\epsilon}^*$. See [63] for a detailed exploration on Lie derivative and compatibility for Hamiltonian operators.

More generally, any Hamiltonian operator P_0 defines a map $d_{P_0} = [P_0, \cdot]$ which is a differential: $d_{P_0}^2 = 0$. The Hamiltonian cohomology, or Lichnerowicz–Poisson cohomology of d_{P_0} measures the presence of operators which are compatible with P_0 and are not the Lie derivative of P_0 , so that they are not deformable back to P_0 .

Another interesting feature of the Lie derivative is that it is useful to find topological hierarchies. Topological hierarchies are related with the theory of Gromov-Witten invariants, the theory of singularities and other topics; for more details, see [23]. In [28]

it was proved that if two Hamiltonian operators A_1 and A_2 of the type (48) fulfilling some further property define a topological hierarchy if and only if they are an *exact Poisson pencil*. The latter condition is, by definition, the existence of a vector field τ such that

$$A_2 = L_\tau A_1, \quad L_\tau A_2 = L_\tau^2 A_1 = 0. \quad (49)$$

Hence, for the above reasons (and more) it is natural to pose the question: given two compatible Hamiltonian operators A_1, A_2 find a vector field τ (if it exists!) such that $A_2 = L_\tau A_1$.

In this paper we will consider two such calculations: one is for the bi-Hamiltonian pair of the KdV-equation, from [19], and the other is for the bi-Hamiltonian pair of the WDVV equation [26] presented as a hydrodynamic-type system [60].

Let us start with the KdV equation; the calculation is in the file `kdv_lieder`. After the initialization, we define the two well-known Hamiltonian operators A_1 and A_2

```
mk_cdiffof(a1,1,{1},1);
for all psi1 let a1(1,1,psi1)=td(psi1,x);

mk_cdiffof(a2,1,{1},1);
for all psi3 let a2(1,1,psi3)=u_x*psi3 + td(psi3,x,3)
+ 2*u*td(psi3,x);
```

and consider the vector field τ as a degree 1 variational multivector `tau` and superfunction `tau_sf`

```
mk_cdiffof(tau,1,{1},1);
for all phi let tau(1,phi) = (- (1/2)*u**2 - (1/2)*u_2x)*phi;

mk_superfun(tau_sf,1,1);
tau_sf(1):= (- (1/2)*u**2 - (1/2)*u_2x)*p;
```

Then, we convert the above operators into the corresponding bivectors

```
conv_cdiffof2superfun(a1,s1);
conv_cdiffof2superfun(a2,s2);

conv_genfun2biv(s1,biv1);
conv_genfun2biv(s2,biv2);
```

and we compute the Schouten bracket $[\tau, A_2]$:

```
schouten_bracket(tau_sf,biv1,l_tau_biv1);
l_tau_biv1(1);
```

The result is not the same as `biv2(1)`; however, they coincide in the variational cohomology, as they should:

euler_df(1_tau_biv1(1) - biv3(1));

{{0},{0}};

the latter being the list of the variational derivatives with respect to even (u) and odd (p) coordinates.

It is known that the simplest WDVV equation: $f_{ttt} = f_{xxt}^2 - f_{xxx}f_{xtt}$ can be rewritten as a hydrodynamic-type system by introducing new variables $a^1 = a = f_{xxx}$, $a^2 = b = f_{xxt}$, $a^3 = c = f_{xtt}$, namely:

$$a_t = b_x, \quad b_t = c_x, \quad c_t = (b^2 - ac)_x. \quad (50)$$

It was proved in [29] that the above system is bi-Hamiltonian, with (compatible) operators

$$A_1 = \begin{pmatrix} -\frac{3}{2}\partial_x & \frac{1}{2}\partial_x a & \partial_x b \\ \frac{1}{2}a\partial_x & \frac{1}{2}(\partial_x b + b\partial_x) & \frac{3}{2}c\partial_x + c_x \\ b\partial_x & \frac{3}{2}\partial_x c - c_x & (b^2 - ac)\partial_x + \partial_x(b^2 - ac) \end{pmatrix}, \quad (51)$$

$$A_2 = \begin{pmatrix} 0 & 0 & \partial_x^3 \\ 0 & \partial_x^3 & -\partial_x^2 a \partial_x \\ \partial_x^3 & -\partial_x a \partial_x^2 & \partial_x^2 b \partial_x + \partial_x b \partial_x^2 + \partial_x a \partial_x a \partial_x \end{pmatrix}. \quad (52)$$

We can rewrite the system and the operators in flat coordinates u^k of the leading term of the operator A_1 (as a flat pseudo-Riemannian contravariant metric):

$$a = u^1 + u^2 + u^3, \quad b = -\frac{1}{2}(u^1 u^2 + u^2 u^3 + u^3 u^1), \quad c = u^1 u^2 u^3, \quad (53)$$

the operator A_1 becomes $A_1^{ij} = K^{ij}\partial_x$, where

$$(K^{ij}) = \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix},$$

and the system (50) takes the form

$$u_t^i = \frac{1}{2}(u^j u^k - u^i u^j - u^i u^k)_x, \quad (54)$$

where i, j, k are three distinct indices. Note that flat coordinates u^i of the first operator A_1 make the expression of the second operator A_2 much more complicated with respect to the initial coordinates a, b, c .

It is proved in [60] that the coordinate expression of τ is

$$\tau = -K^{in} L_n \frac{\partial}{\partial u^i}, \quad \text{where} \quad L_n = \left(\frac{1}{2} G_{nm} u_x^m + R_{nm} u_x^m \right)_x - \frac{1}{2} L_{nsm} u_x^s u_x^m. \quad (55)$$

In the above formula we have:

1. $-\frac{1}{2}L_{nsm}u_x^s u_x^m$ are 3 conserved densities of the system (54) with coordinate expressions (note that $L_{ijk} = L_{ikj}$):

- (a) when $j \neq 1$ and $k \neq 1$ or when $j = k = 1$

$$L_{1jk} = \frac{((u^1 - u^2) + (u^1 - u^3))(u^a - u^b)(u^c - u^d)}{2(u^1 - u^2)^3(u^1 - u^3)^3}, \quad (56)$$

where (a, j, b) , (c, d, k) are triplets of distinct indices with $a < b$, $c < d$;

- (b) when $(1, j, k)$ are a triplet of distinct indices

$$L_{11k} = -\frac{(u^1 - u^k)^2 + (u^1 - u^j)^2}{2(u^1 - u^k)^2(u^1 - u^j)^3}. \quad (57)$$

- (c) when $i \neq 1$ the expressions of L_{ijk} are obtained by a cyclic permutation of the above expressions.

2. G_{nm} is the leading term of the symplectic operator $B_{ij} = -K_{ip}A_2^{pq}K_{qj}$ (see [60]), and it is a pseudo-Riemannian metric; the coordinate expression of the inverse matrix (G^{ij}) is simpler:

$$G^{ii} = -\frac{1}{4}(u^i - u^j)(u^i - u^k), \quad G^{ij} = -\frac{1}{4}(u^i - u^j)^2. \quad (58)$$

3. R_{pm} is a two-form whose expression is

$$R_{ij} = -\frac{1}{3} \left(\frac{1}{(u^i - u^j)(u^i - u^k)} - \frac{1}{(u^j - u^i)(u^j - u^k)} \right), \quad (59)$$

where i, j, k are distinct indices.

Let us describe the computation that leads to directly proving the equality $L_\tau A_1 = A_2$ (file `w3c_lagrep3`). We initialize the jet space with

```
indep_var:={x};
dep_var:={u1,u2,u3};
odd_var:={p1,p2,p3};
total_order:=10;
ncomp:=length(dep_var);
```

We load the third-order operator A_2 as in subsection 4.1, then we set up the change of variables with its Jacobian:

```

a:=u1 + u2 + u3;
a_x:=td(a,x);
b:=-1/2*(u1*u2 + u2*u3 + u3*u1);
b_x:=td(b,x);
c:=u1*u2*u3;
c_x:=td(c,x);
a_uqs:={a,b,c};

matrix jac(ncomp,ncomp);
for i:=1:ncomp do
  for j:=1:ncomp do jac(i,j):=df(part(a_uqs,i),part(dep_var,j));
jacinv:=jac**(-1);

```

We can use the following formula to change coordinates to the operator A_2 :

$$A_2^{ij}(\mathbf{u}) = \frac{\partial u^i}{\partial a^n} A_2^{nm}(\mathbf{a}) \frac{\partial u^j}{\partial a^m}, \quad (60)$$

implemented as

```

mk_cdifop(taa2,1,{3},3);
for all i,j,psi let taa2(i,j,psi)=
  (for h:=1:ncomp sum
    (for k:=1:ncomp sum jacinv(i,h)*aa2(h,k,jacinv(j,k)*psi))
  );

```

Obviously, `taa2` is the operator A_2 in the coordinates u^i . The various tensors G_{mn} , R_{mn} and L_{nmp} can be obtained from the coefficients of the operator `taa2` (see the file `w3c.lagrep3`). The operator `l_n` can be defined in such a way that the expression of `l_n(n)` will be L_n . Then we define the operator A_1 (in coordinates u^i)

```

matrix kap(ncomp,ncomp),kapinv(ncomp,ncomp);
kap:=(1/2)*mat((1,-1,-1),(-1,1,-1),(-1,-1,1));
kapinv:=kap**(-1);

mk_cdifop(taa1,1,{3},3);
for all i,j,psi let taa1(i,j,psi) =
  kap(i,j)*td(psi,x);

```

and vector field τ , with its conversion to an operator and a superfunction

```

mk_cdifop(tau,1,{3},1);
for all i,phi let tau(i,1,phi)=
  (for j:=1:ncomp sum - kap(i,j)*l_n(j))*phi;
mk_superfun(tau_sf,1,1);
tau_sf(1):=for i:=1:ncomp sum tau(i,1,part(odd_var,i));

```

After the conversion of the Hamiltonian operators to bivectors:

```
conv_cdifffun2superfun(taa1,taa1_sf);
conv_cdifffun2superfun(taa2,taa2_sf);
conv_genfun2biv(taa1_sf,biv1);
conv_genfun2biv(taa2_sf,biv2);
```

we can compute the Lie derivative $L_\tau A_1$ as the Schouten bracket $[\tau, A_1]$

```
schouten_bracket(tau_sf,biv1,l_tau_biv1);
```

and check that the result coincides with `biv2` in the variational cohomology:

```
euler_df(l_tau_biv1(1) - biv2(1));
```

Beware: the computation takes 18 hours and 8GB RAM on a (not too fast) compute server.

5 Problem. Repeat the above computation for the bi-Hamiltonian pair of the WDVV equation in the case $N = 4$ [59]. Can the computation be successfully carried out for the oriented associativity equation [15]?

6 Remark. A few simple examples of computations of Schouten bracket can be found in the CDE manual (which is part of the official Reduce manual).

We did not show any calculation of Schouten bracket between operators in more than one independent variable due to space constraints. CDE has this capability, the interested Reader can find one simple example (provided by Casati) in the CDE manual. More non-trivial examples, with applications to Hamiltonian and bi-Hamiltonian cohomology, are computed (also by CDE) in the recent paper [14].

5 Conclusions

In this paper we deliberately did not deal with nonlocal operators, i.e., operators which contain expressions with ∂_x^{-1} . At the moment, the only publicly available software for computations of Schouten bracket between nonlocal operators is the Maple package JET, but it has some limitations: indeed, it cannot always simplify expressions that contain ∂_x^{-1} . Recently, an algorithm for bringing such expression to a canonical form has been introduced [16], and we plan to implement it as an additional module of CDE in the near future. An extension of the odd variable formalism for the Schouten bracket of nonlocal operators is in progress [45].

An interesting perspective of application of CDE could be in theoretical physics, wherever supermanifolds play a role. Indeed, CDE can do every calculation on jets of superbundles (see [39, 42, 47]). This means, for example, Euler–Lagrange expressions of superfunctions, symmetries and conserved quantities of super-differential equations, etc.. Hence, it could be of help in complex computations in supersymmetric mechanics and field theories.

Acknowledgments

I would like to thank P. Kersten and A. Norman for their invaluable Reduce support. Thanks are due to M. Casati, B.A. Dubrovin, E.V. Ferapontov, A. Fordy, J.S. Krasil'shchik, M.V. Pavlov, A. Sergyeyev, A.M. Verbovetsky for scientific discussions and motivation to write and improve the software.

Finally, I would like to thank A. Falconieri, the system administrator of the workstation `sophus2` of the Department of Mathematics and Physics of the Università del Salento.

This research has been funded by the Dept. of Mathematics and Physics “E. De Giorgi” of the Università del Salento, Istituto Naz. di Fisica Nucleare IS-CSN4 *Mathematical Methods of Nonlinear Physics*, GNFM of Istituto Nazionale di Alta Matematica.

References

- [1] M.J. Ablowitz and P.A. Clarkson. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*. Number 149 in London Mathematical Society lecture note series. Cambridge University Press, 1991.
- [2] I. Anderson. *Integrable Systems Tools*, 2017. Maple package available at <http://digitalcommons.usu.edu/dg/>.
- [3] A. M. Astashov and A. M. Vinogradov. On the structure of Hamiltonian operators in field theory. *J. Geom. Phys.*, 3:263–287, 1986.
- [4] M.A. Ayari and V. Hussin. GLie; a MAPLE program for lie supersymmetries of Grassmann-valued differential equations. *Comput. Phys. Comm.*, 100:157, 1997.
- [5] D. Baldwin and W. Hereman. A symbolic algorithm for computing recursion operators of nonlinear partial differential equations. *International Journal of Computer Mathematics*, 87(5):1094–1119, 2010.
- [6] A. Barakat, A. De Sole, and V. G. Kac. Poisson vertex algebras in the theory of Hamiltonian equations. *Japan. J. Math.*, 4:141–252, 2009.
- [7] M. Barakat. Jets. a Maple-package for formal differential geometry. In *Computer algebra in scientific computing*, pages 1–12. Springer, 2001.
- [8] H. Baran, I.S. Krasil'shchik, O.I. Morozov, and P. Vojčák. Symmetry reductions and exact solutions of Lax integrable 3-dimensional systems. *J. Nonlinear Math. Phys.*, 21(4):643–671, 2014. arXiv:1407.0246.

- [9] H. Baran, I.S. Krasil'shchik, O.I. Morozov, and P. Vojčák. Five-dimensional Lax-integrable equation, its reductions and recursion operator. *Lobachevskii J. Math.*, 36(3):225–233, 2015.
- [10] H. Baran, I.S. Krasil'shchik, O.I. Morozov, and P. Vojčák. Coverings over Lax integrable equations and their nonlocal symmetries. *Theoret. and Math. Phys.*, 188(3):1273–1295, 2016. arXiv:1507.00897.
- [11] H. Baran and M. Marvan. *Jets. A software for differential calculus on jet spaces and diffieties*. Silesian University in Opava, 2003 and 2010. <http://jets.math.slu.cz/>.
- [12] H. Baran and M. Marvan. On integrability of Weingarten surfaces: A forgotten class. *J. Phys. A*, 42:404007, 2009.
- [13] A. V. Bocharov, V. N. Chetverikov, S. V. Duzhin, N. G. Khor'kova, I. S. Krasil'shchik, A. V. Samokhin, Yu. N. Torkhov, A. M. Verbovetsky, and A. M. Vinogradov. *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics*. Monograph. Amer. Math. Soc., 1999. I. S. Krasil'shchik and A. M. Vinogradov editors.
- [14] M. Casati. Higher order dispersive deformations of multidimensional Poisson brackets of hydrodynamic type. arXiv:1710.08175, 2017.
- [15] M. Casati, E.V. Ferapontov, M.V. Pavlov, and R.F. Vitolo. On a class of third-order nonlocal Hamiltonian operators. *J. Geom. Phys.*, 2019. DOI: <https://doi.org/10.1016/j.geomphys.2018.10.018>, <https://arxiv.org/abs/1805.00746>.
- [16] M. Casati, P. Lorenzoni, and R. Vitolo. Three computational approaches to weakly nonlocal poisson brackets. <https://arxiv.org/abs/1903.08204>, 2019.
- [17] M. Casati and D. Valeri. MasterPVA and WAlg: Mathematica packages for poisson vertex algebras and classical affine \mathcal{W} -algebras. *Boll. Un. Mat. Ital.*, 11(4):503–531, 2018. <https://arxiv.org/abs/1603.05028>.
- [18] E. Cheb-Terrab. Symbolic Computing with Grassman Variables. <https://arxiv.org/abs/hep-th/9510226>, 1995.
- [19] L. Degiovanni, F. Magri, and V. Sciacca. On deformation of Poisson manifolds of hydrodynamic type. *Comm. Math. Phys.*, 253(1):1–24, 2005.
- [20] I. Ya. Dorfman. *Dirac Structures and Integrability of Nonlinear Evolution Equations*. John Wiley & Sons, 1993.

- [21] B. Dubrovin. Flat pencils of metrics and Frobenius manifolds. In M.-H. Saito, Y. Shimizu, and K. Ueno, editors, *Proceedings of 1997 Taniguchi Symposium “Integrable Systems and Algebraic Geometry”*, pages 42–72. World Scientific, 1998. https://people.sissa.it/~dubrovin/bd_papers.html.
- [22] B. Dubrovin and Y. Zhang. Bi-Hamiltonian hierarchies in 2D topological field theory at one-loop approximation. *Comm. Math. Phys.*, 198:311–361, 1998.
- [23] B. Dubrovin and Y. Zhang. Normal forms of integrable PDEs, Frobenius manifolds and Gromov–Witten invariants,. [math.DG/0108160](https://arxiv.org/abs/math/0108160), 2001.
- [24] B. A. Dubrovin, I. M. Krichever, and S. P. Novikov. Integrable systems. I. In *Dynamical Systems IV*, volume 4 of *Encyclopaedia of Mathematical Sciences*, pages 173–280. Springer-Verlag, Berlin, 2 edition, 2001.
- [25] B. A. Dubrovin and S. P. Novikov. Poisson brackets of hydrodynamic type. *Soviet Math. Dokl.*, 30:651–654, 1984.
- [26] B.A. Dubrovin. Geometry of 2d topological field theories. *Lect. Notes Math.*, 1620:120–348, 1996.
- [27] B.A. Dubrovin and S.P. Novikov. Hamiltonian formalism of one-dimensional systems of hydrodynamic type and the bogolyubov-whitham averaging method. *Soviet Math. Dokl.*, 27(3):665–669, 1983.
- [28] G. Falqui and P. Lorenzoni. Exact poisson pencils, τ -structures and topological hierarchies. *Phys. D*, 241:2178–2187, 2012.
- [29] E.V. Ferapontov, C.A.P. Galvao, O. Mokhov, and Y. Nutku. Bi-hamiltonian structure of equations of associativity in 2-d topological field theory. *Comm. Math. Phys.*, 186:649–669, 1997.
- [30] E.V. Ferapontov and O.I. Mokhov. On the Hamiltonian representation of the associativity equations. In I.M. Gelfand and A.S. Fokas, editors, *Algebraic aspects of integrable systems: in memory of Irene Dorfman*, pages 75–91. Birkhäuser Boston, 1996.
- [31] E.V. Ferapontov, M.V. Pavlov, and R.F. Vitolo. Projective-geometric aspects of homogeneous third-order hamiltonian operators. *J. Geom. Phys.*, 85:16–28, 2014. DOI:10.1016/j.geomphys.2014.05.027.
- [32] E.V. Ferapontov, M.V. Pavlov, and R.F. Vitolo. Towards the classification of homogeneous third-order Hamiltonian operators. *Int. Math. Res. Not.*, 22:6829–6855, 2016.

- [33] E.V. Ferapontov, M.V. Pavlov, and R.F. Vitolo. Systems of conservation laws with third-order Hamiltonian structures. *Lett. Math. Phys.*, 108(6):1525–1550, 2018. <https://arxiv.org/abs/1703.06173>.
- [34] I. M. Gel'fand and I. Ya. Dorfman. Hamiltonian operators and algebraic structures related to them. *Funct. Anal. Appl.*, 13:248–262, 1979.
- [35] E. Getzler. A Darboux theorem for Hamiltonian operators in the formal calculus of variations. *Duke Math. J.*, 111:535–560, 2002.
- [36] U. Hartmann and E.D. Davis. epic Grass - symbolic calculations with anticommuting variables. *Comput. Phys. Comm.*, 54(2–3):353–369, 1989.
- [37] A.C. Hearn. *Reduce*. <http://reduce-algebra.sourceforge.net/>, version 3.8 edition, 2004. Computer algebra system, currently in development after that it has been released in 2008 as free software at Sourceforge. The manual is available at the website.
- [38] S. Igonin, A. Verbovetsky, and R. Vitolo. On the formalism of local variational differential operators. Memorandum 1641, Faculty of Mathematical Sciences, University of Twente, The Netherlands, 2002.
- [39] S. Igonin, A. Verbovetsky, and R. Vitolo. Variational multivectors and brackets in the geometry of jet spaces. In *Symmetry in Nonlinear Mathematical Physics. Part 3*, pages 1335–1342. Institute of Mathematics of NAS of Ukraine, Kiev, 2003.
- [40] J. Kalayci and Y. Nutku. Bi-Hamiltonian structure of a WDVV equation in $2d$ topological field theory. *Phys. Lett. A*, 227:177–182, 1997.
- [41] J. Kalayci and Y. Nutku. Alternative bi-Hamiltonian structures for WDVV equations of associativity. *J. Phys. A*, 31:723–734, 1998.
- [42] P. Kersten, I. Krasil'shchik, and A. Verbovetsky. Hamiltonian operators and ℓ^* -coverings. *J. Geom. Phys.*, 50:273–302, 2004.
- [43] P. Kersten, I. Krasil'shchik, A. Verbovetsky, and R. Vitolo. Hamiltonian structures for general PDEs. In B. Kruglikov, V. V. Lychagin, and E. Straume, editors, *Differential equations: Geometry, Symmetries and Integrability. The Abel Symposium 2008*, pages 187–198. Springer-Verlag, 2009.
- [44] I. S. Krasil'shchik and P. H. M. Kersten. *Symmetries and Recursion Operators for Classical and Supersymmetric Differential Equations*. Kluwer, 2000.

- [45] J. Krasil'shchik and A. Verbovetsky. Nonlocal jacobi identity: a geometric approach. Talk presented at the conference in honour of the 70th birthday of J. Krasil'shchik, SISSA, Trieste. See <https://gdeq.org/files/VerbovetskyTrieste2018slides.pdf>, 2018.
- [46] J. Krasil'shchik, A. Verbovetsky, and R. Vitolo. *The symbolic computation of integrability structures for partial differential equations*. Texts and Monographs in Symbolic Computation. Springer, 2018. ISBN 978-3-319-71654-1; see http://gdeq.org/Symbolic_Book for downloading program files that are discussed in the book.
- [47] J. Krasil'shchik and A. M. Verbovetsky. Geometry of jet spaces and integrable systems. *J. Geom. Phys.*, 61:1633–1674, 2011. arxiv:1002.0077.
- [48] P. Lorenzoni, A. Savoldi, and R. Vitolo. Bi-Hamiltonian systems of KdV type. *J. Phys. A*, 51(4):045202, 2018.
- [49] F. Magri. A simple model of the integrable Hamiltonian equation. *J. Math. Phys.*, 19:1156–1162, 1978.
- [50] A.G. Meshkov. Tools for symmetry analysis of PDEs. *Differential Equations and Control Processes*, 1, 2002. <http://www.math.spbu.ru/diffjournal/>.
- [51] O. Mokhov. Compatible Dubrovin—Novikov Hamiltonian operators, Lie derivative, and integrable systems of hydrodynamic type. *Theoret. and Math. Phys.*, 133(2):1557–1564, 2002.
- [52] O.I. Mokhov. Local third-order poisson brackets. *Russian Math. Surveys*, 40(5):233–234, 1985.
- [53] O.I. Mokhov. Hamiltonian differential operators and contact geometry. *Funct. Anal. Appl.*, 21(3):217–223, 1987.
- [54] O.I. Mokhov. Symplectic and Poisson geometry on loop spaces of smooth manifolds and integrable equations. In S.P. Novikov and I.M. Krichever, editors, *Reviews in mathematics and mathematical physics*, volume 11, pages 1–128. Harwood academic publishers, 1998.
- [55] A.C. Norman and R. Vitolo. *Inside Reduce*, 2014. part of the official Reduce documentation, included in the source code of Reduce. See also <http://reduce-algebra.sourceforge.net/lisp-docs/insidereduce.pdf>.
- [56] P. Olver. Darboux' theorem for hamiltonian differential operators. *J. Differential Equations*, 71:10–33, 1988.

- [57] P.J. Olver. *Applications of Lie Groups to Differential Equations*. Springer-Verlag, 2nd edition, 1993.
- [58] P.J. Olver and P. Rosenau. Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support. *Phys. Rev. E (3)*, 53:1900–1906, 1996.
- [59] M.V. Pavlov and R.F. Vitolo. On the bi-Hamiltonian geometry of the WDVV equations. *Lett. Math. Phys.*, 105(8):1135–1163, 2015.
- [60] M.V. Pavlov and R.F. Vitolo. Remarks on the Lagrangian representation of bi-Hamiltonian equations. *J. Geom. Phys.*, 113C:239–249, 2017.
- [61] K. Peeters. A field-theory motivated approach to symbolic computer algebra. *Comput. Phys. Comm.*, 176:550–558, 2006. <https://arxiv.org/abs/cs.sc/0608005>.
- [62] G.V. Potemin. On third-order Poisson brackets of differential geometry. *Russian Math. Surveys*, 52:617–618, 1997.
- [63] A. Sergyeyev. A simple way of making a Hamiltonian system into a bi-Hamiltonian one. *Acta Appl. Math.*, 83(1–2):183–197, 2004. DOI:10.1023/B:ACAP.0000035597.06308.8a.
- [64] R. Vitolo. Variational sequences. In D. Krupka and D. J. Saunders, editors, *Handbook of global analysis*, pages 1115–1163, 1217. Elsevier, 2008.
- [65] T. Wolf. An efficiency improved program LIEPDE for determining Lie-symmetries of PDEs. In *Proceedings of Modern Group Analysis: advanced analytical and computational methods in mathematical physics*, pages 1–20. Kluwer, 1993.
- [66] T. Wolf and A. Brand. Investigating DEs with CRACK and related programs. *SIGSAM Bulletin, Special Issue*, pages 1–8, 1995.
- [67] T. Wolf and A. Brand. *CRACK, user guide, examples and documentation*, 2006. http://lie.math.brocku.ca/Crack_demo.html.
- [68] V. E. Zakharov, editor. *What is integrability?* Springer-Verlag, Berlin, 1991.