Remarks on the Lagrangian representation of bi-Hamiltonian equations

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Abstract

The Lagrangian representation of multi-Hamiltonian PDEs has been introduced by Y. Nutku and one of us (MVP). In this paper we focus on systems which are (at least) bi-Hamiltonian by a pair A_1, A_2, where A_1 is a hydrodynamic-type Hamiltonian operator. We prove that finding the Lagrangian representation is equivalent to finding a generalized vector field τ such that $A_2 = L_\tau A_1$. We use this result in order to find the Lagrangian representation when A_2 is a homogeneous third-order Hamiltonian operator, although the method that we use can be applied to any other homogeneous Hamiltonian operator. As an example we provide the Lagrangian representation of a WDVV hydrodynamic-type system in 3 components.

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1 Introduction

Integrable systems of nonlinear PDEs in $1+1$ dimensions (or, equivalently, in 2 independent variables) have a particularly rich structure. Evolutionary systems of the form

$$u^i_t = f^i(u^k, u^k_x, u^k_{xx}, \ldots) \tag{1}$$

can be given one or more Hamiltonian formulation

$$u^i_t = f^i(u^k, u^k_x, u^k_{xx}, \ldots) = A^{ij}_\alpha \frac{\delta \mathbf{H}_\alpha}{\delta u^j}, \quad \text{(no sum in } \alpha), \quad i = 1, \ldots, n \tag{2}$$

through one or more Hamiltonian operators A_α. Here, α is an index running from 1 to M, $\mathbf{H}_\alpha$ are Hamiltonian densities and $\delta/\delta u^j = (-1)^{|\sigma|}\partial_\sigma\, \partial/\partial u^i_\sigma$ are variational derivatives (σ is a multi-index). The requirement of A_α being Hamiltonian amounts at A_α being a formally skew-adjoint differential operator in total derivatives ∂_σ and $[A_\alpha, A_\alpha] = 0$. The last operator is the Schouten bracket, see [5] for a definition. Hamiltonian operators map characteristic vectors of conservation laws into generating functions of (generalized) symmetries [5, 25].

By a classical result by F. Magri (see [5] and references therein), the integrability of the system (2) is ensured by the existence of at least 2 Hamiltonian operators that are compatible: $[A_1, A_2] = 0$. This yields a sequence of commuting conserved quantities and symmetries and, eventually, leads to integration through the inverse scattering method.

An interesting feature of bi-Hamiltonian systems ($M = 2$) or multi-Hamiltonian systems ($M > 2$) was first pointed out in [24]. Suppose that the Hamiltonian operator A_β is invertible. Then the operators $R^\alpha_\beta = A_\alpha A_\beta^{-1}$ applied to eq. (2) yield, after potential substitution $u^i = \varphi^i_x$, systems of PDEs which admit local Lagrangians in the new variables φ^i. If all the operators A_α are invertible there will be $2M - 1$ new systems. This is the *Lagrangian representation* of the system (2). It is clear that having the possibility to represent a system of PDEs and its hierarchy of symmetries by Lagrangian PDEs is mathematically interesting. This possibility was considered by several authors (see, *e.g.*,[23]), and was systematically linked to the multi-Hamiltonian property in [24]. Several examples have been considered so far: KdV, polytropic gas dynamics, Kaup–Boussinesq, Kaup–Broer, Boussinesq, NLS [24].

In this paper we will consider evolutionary systems of PDEs which are at least bi-Hamiltonian by a pair of Hamiltonian operators A_1, A_2:

$$u^i_{t^k} = A_2^{is}\frac{\delta \mathbf{H}_k}{\delta u^s} = A_1^{is}\frac{\delta \mathbf{H}_{k+1}}{\delta u^s}, \tag{3}$$

where for $k = 0$ we have the initial system of the hierarchy of commuting flows. We make the assumption that the first Hamiltonian operator A_1 is a first-order Dubrovin–Novikov homogeneous Hamiltonian operator

$$A_1^{ij} = g^{ij}\partial_x + b^{ij}_k u^k_x, \tag{4}$$

where g^{ij}, b^{ij}_k are functions of the field variables (u^h) and homogeneity is meant with respect to the grading that assigns the degree 1 to ∂_x. Such operators are quite common among the bi-Hamiltonian systems. Moreover, they are easily invertible: in the non-degenerate case $\det(g^{ij}) \neq 0$ there is always a transformation $\bar{u}^i = \bar{u}^i(u^j)$ such that $A_1 = K^{ij}\partial_x$, where K^{ij} is a constant symmetric matrix, hence $A_1^{-1} = (K^{ij})^{-1}\partial_x^{-1}$.

Bi-Hamiltonian systems of the above type are known to have the following properties (besides admitting a Lagrangian representation).

1. They admit a nonlocal *symplectic operator* $B = A_1^{-1}A_2A_1^{-1}$ [5, Proposition 7.9]. This operator becomes local after the above potential substitution, and the above system (3) can be rewritten as

$$B_{im}\varphi^m_{t^k} = \frac{\delta \mathbf{H}_{k+2}}{\delta \varphi^i}, \tag{5}$$

The above equation is evidently Lagrangian; its *Lagrangian representation* is a Lagrangian for the above equation of the form

$$L_n\varphi^n_{t^k} - h_{k+2}, \tag{6}$$

where h_{k+2} is the function that defines the Hamiltonian density $\mathbf{H}_{k+2}$ (which is a quantity that can be computed) and (L_n) is a vector function of φ^i, φ^i_x, φ^i_{xx}, ..., called the *characteristic function* of the Lagrangian representation, to be determined.

2. The second Hamiltonian operator is the Lie derivative of the first one: $A_2 = L_\tau A_1$, where τ is a generalized vector field (see [5, Proposition 7.9] or, more generally [3, 15, 10]). According with a modern terminology [10, 21], A_2 is a *trivial infinitesimal deformation* of A_1.

Clearly, the difficult part of finding the Lagrangian representation of the equation (10) is finding (L_n), as h_{k+2} can be found from the initial equation by bi-Hamiltonian recursion with usual methods. We stress that the Lagrangian representation is *local*, as there are topological obstructions in finding global Lagrangians, see the discussion below.

In this paper we will prove the following:

Theorem. *Finding the Lagrangian representation* (12) *of the integrable hierarchy* (3) *is equivalent to finding the generalized vector field* τ *such that* $A_2 = L_\tau A_1$.

More precisely, we will prove that the vector function $\psi = -(L_n)$ is nothing but a potential of the symplectic form B with respect to the differential e_1 of the variational sequence [2, 5, 25, 30]. Variational sequences have been introduced as the analogue of the de Rham sequence for the calculus of variations. Let us summarize the features of this topic that are relevant to this paper, see [2, 5, 25, 30] for much deeper insight. Variational sequences can be schematically represented by the sequence of differential operators

$$\dots \xrightarrow{D} \text{Lagrangians} \xrightarrow{\mathcal{E}} \text{Variational 1-forms} \xrightarrow{\mathcal{H}} \text{Variational 2-forms} \dots \tag{7}$$

where D is the total divergence, $\mathcal{E}$ is the Euler-Lagrange operator, $\mathcal{H}$ is the Helmholtz operator any two consequent operations are identically zero. Variational 1-forms are covector-valued densities ψ which define systems of PDEs of the form $\psi = 0$. If $H(\psi) = 0$ then, locally, the system $\psi = 0$ is Lagrangian: locally there exists a Lagrangian density L such that $\mathcal{E}(L) = \psi$, and L can be computed by the Volterra homotopy operator [2, 5, 25, 30].

The spaces of variational k-forms (variational 0-forms are Lagrangians) form sheaves, and the variational sequence turns out to be a locally exact sequence of sheaves whose differentials are all denoted by e_1. Then, a variational 2-form B is symplectic iff $e_1(B) = 0$. If this is true, by local exactness there exists a variational 1-form ψ (which is in general only locally defined) such that $H(\psi) = B$. In the proof of the above Theorem it will be shown that $\tau = A_1(\psi)$, therefore the Lagrangian representation will have the same local (or global) character as ψ. There are well-known topological obstructions to the global exactness of the variational sequence [2, 5, 25, 30].

The above results hold in a differential-geometric setting, *i.e.* for $\mathcal{C}^\infty$ forms and vector fields, despite the fact that they have initially been introduced having polynomial categories in mind [5].

With the above ideas in mind we found new nontrivial examples of Lagrangian representation, besides those that were discussed so far, in the class of bi-Hamiltonian systems where A_2 is a third-order homogeneous Hamiltonian operator. In this class the homotopy operator of the variational sequence allows us to choose a vector function (L_n) which is: 1 - homogeneous, since the homotopy operator preserves homogeneity, and 2 - dependent on third-order derivatives, since the homotopy operator preserves the order of derivatives.

However, we had to overcome an obstacle. We are forced to compute in flat coordinates of the first Hamiltonian operator A_1, as this makes A_1 easily invertible. In these coordinates the coefficients of the homogeneous operator A_2 yield non-removable singularities in the homotopy operator. So, we had to develop a different method that, in principle, can be used with any homogeneous operator A_2 of arbitrary order. We proved the following:

Theorem. *There exists a Lagrangian representation* (6) *where L_n is a homogeneous polynomial of derivatives of degree 2 if and only if the integrability condition $dT = 0$ is fulfilled, where T is a three-form defined in* (23)*. In this case we have*

$$L_n = \left(\frac{1}{2}G_{nm}u^m_x + R_{nm}u^m_x\right)_x - \frac{1}{2}L_{nsm}u^s_x u^m_x \qquad (8)$$

where (u^i) are flat coordinates of g, G_{nm} is the leading coefficient of the symplectic operator B, $L_{nsm}u^s_x u^m_x$ are n conservation law densities of the initial system of PDEs and R_{nm} is a skew-symmetric tensor.

In our method we made a minimal order ansatz for the vector function (L_n): it can be a homogeneous polynomial of order not lower than 2. This simplifies computations at the cost of leading us to the integrability condition $dT = 0$.

The above results have been successfully used to compute the Lagrangian representation for the WDVV system of hydrodynamic type (30) for which the bi-Hamiltonian nature is known [11]. The procedure can be applied with equal ease to the bi-Hamiltonian systems [17, 18, 26], thus providing a wide range of examples. In all the above cases the integrability condition is fulfilled and it is conjectured that the condition is a consequence of the fact that B is a symplectic operator. See the Conclusions for more details.

Symbolic computations were done in Reduce, a free Computer Algebra System, using the package CDE, developed by the author of this paper [31]. Interested readers are warmly invited to contact the author for question on any aspect of the computations or to get software and/or mathematical expressions of quantities that would be unpractical to include in the paper.

2 Lagrangian representation and triviality of A_2

In this section we will recall the construction of the Lagrangian representation of a bi-Hamiltonian system in the case when A_1 is a first-order Dubrovin–Novikov homogeneous Hamiltonian operator. Then we will show the relationship between the (local) existence of the Lagrangian representation and the (local) existence of a generalized vector field τ such that $A_2 = L_\tau A_1$.

Let us recall how the Lagrangian representation is defined in our framework. We work in flat coordinates (u^i) of A_1, so that $A_1 = K^{ij}\partial_x$. After the potential substitution $u^i = \varphi^i_x$ the integrable hierarchy (3) becomes

$$\varphi^i_{xt^k} = -A_2^{is}\partial_x^{-1}\frac{\delta \mathbf{H}_k}{\delta\varphi^s} = -K^{is}\frac{\delta \mathbf{H}_{k+1}}{\delta\varphi^s}. \tag{9}$$

Then we take two copies of this integrable hierarchy (here $M_{ij}K^{js} = \delta^s_i$)

$$-M_{im}\varphi^m_{xt^k} = M_{im}A_2^{ms}\partial_x^{-1}\frac{\delta \mathbf{H}_k}{\delta\varphi^s} = \frac{\delta \mathbf{H}_{k+1}}{\delta\varphi^i},$$
$$-M_{im}\varphi^m_{xt^{k+1}} = M_{im}A_2^{ms}\partial_x^{-1}\frac{\delta \mathbf{H}_{k+1}}{\delta\varphi^s} = \frac{\delta \mathbf{H}_{k+2}}{\delta\varphi^i},$$

which leads to two series of recursive relationships

$$\frac{\delta \mathbf{H}_{k+1}}{\delta\varphi^i} = M_{ip}A_2^{pq}\partial_x^{-1}\frac{\delta \mathbf{H}_k}{\delta\varphi^q}, \quad \varphi^i_{xt^{k+1}} = A_2^{ip}M_{pq}\varphi^q_{t^k}.$$

Thus

$$\frac{\delta \mathbf{H}_{k+2}}{\delta\varphi^i} = -M_{im}\varphi^m_{xt^{k+1}} = -M_{im}A_2^{mp}M_{pq}\varphi^q_{t^k}.$$

This means that the above equations are nothing but Euler–Lagrange equations

$$B_{im}\varphi^m_{t^k} = \frac{\delta \mathbf{H}_{k+2}}{\delta\varphi^i}, \tag{10}$$
$$B_{ij} = -M_{ip}A_2^{pq}M_{qj} \tag{11}$$

where $B = (B_{ij})$ is a local differential operator. It is not difficult to prove that B is a symplectic operator [5].

We observe that the system (9) is Lagrangian with respect to the action density $\frac{1}{2}M_{ij}\varphi^i_x\varphi^j_{t^k} - h_{k+1}$, where $H_k = \int h_k dx$. The equation (10) is Lagrangian too; this fact is characteristic of bi-Hamiltonian systems and was systematically investigated in [24]. *The Lagrangian representation of the bi-Hamiltonian system* (3) is the Lagrangian of the system (10), which is of the type

$$L_n\varphi^n_{t^k} - h_{k+2}. \tag{12}$$

Now, we would like to prove one of the main results of this paper, *ie* the fact that the existence of a Lagrangian representation is equivalent to the triviality of A_2 as an infinitesimal deformation of A_1, or $A_2 = L_\tau A_1$.

Let us recall the notion of Lie derivative for Hamiltonian operators. We denote the derivative coordinates $u^i_{x\cdots x}$ by u^i_σ, where the index σ is the order of the x-derivative. Given any vector function $F = (F_i)$, $i = 1,\ldots,n$ we denote the linearization (or Fréchet derivative) of F and its formal adjoint, respectively acting on a generalized vector field $\tau = \tau^i\partial/\partial u^i$ and on a covector $\psi = \psi_j du^j$, where $\tau^i = \tau^i(u^k_\sigma)$ and $\psi_j = \psi_j(u^k_\sigma)$, by

$$\ell_F(\tau) = \frac{\partial F^k}{\partial u^i_\sigma}\partial_\sigma\tau^i, \quad \ell^*_F(\psi) = (-1)^{|\sigma|}\partial_\sigma\left(\frac{\partial F^k}{\partial u^i_\sigma}\psi_k\right).$$

The above definition extends also to matrix differential operators: if $A(X) = A^{i\sigma}_j\partial_\sigma X^j$, then we have the differential operator in two arguments $\ell_{A,X}(Y) = (\partial A^{i\sigma}_j/\partial u^k_\mu)\partial_\sigma X^j\partial_\mu Y^k$. Then

$$L_\tau A_1(\psi) = -[A_1,\tau](\psi) = \ell_{A_1,\psi}(\tau) - \ell_\tau(A_1(\psi)) - A_1(\ell^*_\tau(\psi)),$$

where ψ is an arbitrary vector-valued density and $[A_1,\tau]$ is the Schouten bracket (see, for example, [5]).

It is known [5, Theorem 7.9] that, if A_1 and A_2 is a pair of compatible Hamiltonian operators where A_1 is invertible, then $B = A_1^{-1}A_2A_1^{-1}$ is a symplectic operator. Moreover, if ψ is a potential of B in the variational sequence, *i.e.* $e_1(\psi) = B$, then the vector field $\tau = A_1(\psi)$ yields $A_2 = L_\tau A_1$. This fact can be regarded as a direct consequence of a more general fact, namely, the vanishing of the Lichnerowicz–Poisson cohomology (which is defined by the differential $d_1 = [A_1,\cdot]$) for first-order homogeneous operators. See [15, 3], or [10, Lemma 2.4.20] for a more recent proof. See also [29] for a complete discussion about the problem of characterizing which operators of the form $L_\tau A_1$ are Hamiltonian.

There is a deep link between the Lagrangian representation, the symplectic form B and the fact that A_2 can be expressed as the Lie derivative of A_1.

Theorem 1. *The systems* (10) *admit a Lagrangian* $L_n\varphi^n_{t^k} - h_{k+2}$ *where* $\psi = -(L_n)$ *is a potential of the symplectic form in the variational sequence:* $e_1(\psi) = B$*. Moreover, we have*

$$A_2 = L_\tau A_1, \quad \textit{where} \quad \tau = A_1(\psi) = -K^{in}L_n\frac{\partial}{\partial u^i}. \tag{13}$$

Proof. Any symplectic form admits a (local) potential $\psi = -(L_n)$ with respect to the differential e_1 in the variational sequence [2, 5, 30]. This means in coordinates

$$B_{in} = -\left(\frac{\partial L_i}{\partial\varphi^n_\sigma}\partial_\sigma - (-1)^\sigma\partial_\sigma\frac{\partial L_n}{\partial\varphi^i_\sigma}\right), \tag{14}$$

Then, it is easy to prove that

$$\frac{\delta(L_n\varphi^n_t)}{\delta\varphi^i} = (-1)^\sigma\partial_\sigma\left(\frac{\partial L_n}{\partial\varphi^i_\sigma}\varphi^n_t\right) - \frac{\partial L_i}{\partial\varphi^n_\sigma}\partial_\sigma\varphi^n_t = B_{in}\varphi^n_t, \tag{15}$$

so that $L_n\varphi^n_{t^k} - h_{k+2}$ is the Lagrangian of (10).

Let us recall that $\ell_\psi = -\partial L_i/\partial\varphi^n_\sigma\partial_\sigma$. Then, eq. (14) can be rewritten as

$$B = \ell_\psi - \ell^*_\psi \tag{16}$$

in potential variables φ^i_x. Changing coordinates to u^i yields the following changes of variables:

$$\ell_\psi(\varphi^i_x, \varphi^i_{xx}, \ldots) = \ell_\psi(u^i, u^i_x, \ldots)\circ\partial_x, \quad \ell^*_\psi(\varphi^i_x, \varphi^i_{xx}, \ldots) = -\partial_x\circ\ell^*_\psi(u^i, u^i_x, \ldots).$$

After multiplying (11) to the left and right by K we have, in coordinates (u^i),

$$A_2^{ij} = -K^{ih}(\ell_\psi\circ\partial_x + \partial_x\circ\ell^*_\psi)_{hk}K^{kj} \tag{17}$$

We would like to prove that $A_2 = L_\tau A_1$. A natural candidate for τ is $\tau = K\psi$: we have $\ell_{A_1,\psi}(\tau) = 0$ because A_1 has constant coefficients, and $\ell_\tau = K\ell_\psi$, so that expanding the definition of the operator $L_\tau A_1$ by computing it on a covector field ξ we have:

$$\begin{aligned} L_\tau A_1(\xi) &= -K\ell_\psi(A_1(\xi)) - A_1((K\ell_\psi)^*(\xi)) \\ &= -K^{ih}(\ell_\psi\circ\partial_x + \partial_x\circ\ell^*_\psi)_{hk}K^{kj} \\ &= A_2(\xi). \end{aligned}$$

□

The above Theorem has the following straightforward consequence.

Corollary 2. *Finding the Lagrangian representation* (12) *of the integrable hierarchy* (3) *is equivalent to finding the generalized vector field* τ *such that* $A_2 = L_\tau A_1$.

Remark 3. *The vector function* $\psi = -(L_n)$ *depends on derivatives of* (φ^i) *which are of the same order of those appearing in* A_2 *(in potential coordinates). This is due to the fact that the homotopy operator of the variational sequence does not change the order of derivatives. Theorems and conjectures about minimising the order in inverse problems of the calculus of variations are discussed in [30].*

3 Lagrangian representation for homogeneous bi-Hamiltonian pairs

It is now clear that the explicit expression of τ also yields the Lagrangian representation of the given bi-Hamiltonian system of PDEs.

Following [5, 3, 10], τ can be computed through ψ, and ψ can be computed using the homotopy operator in the variational sequence. The vector field τ plays the role of the *infinitesimal deformation* in the perturbative approach of [10]. The approach works in the simplest examples like the KdV equation [3]; however, if the range of examples is extended to even just slightly more complicated examples the homotopy operator approach is no longer effective. Indeed the integrand in the homotopy operator is singular for $t = 0$ in many examples (see Section 4). Such singularities are not removable by a shift: the shift operator does not bring total divergencies into total divergencies, hence shifting does not preserve the homotopy operator in the variational sequence (see [5, p. 65]).

In this Section we propose an alternative approach to the computation of τ when A_2 is a homogeneous operator of Dubrovin–Novikov type [9]. The approach will be developed in details when A_2 is of the third order. Indeed, there is a huge family of examples of systems of PDEs which are bi-Hamiltonian with respect to a pair of homogeneous operators A_1 and A_2, with A_1 of the first order and A_2 of the third order, see Section 4. However, our methods easily extend to homogeneous operators of arbitrary order.

Under the above hypotheses A_2 has the form

$$A_2^{ij} = g_2^{ij}(\mathbf{u})\partial_x^3 + b_{2k}^{ij}(\mathbf{u})u_x^k\partial_x^2 + [c_{2k}^{ij}(\mathbf{u})u_{xx}^k + c_{2km}^{ij}(\mathbf{u})u_x^k u_x^m]\partial_x + d_{2k}^{ij}(\mathbf{u})u_{xxx}^k + d_{2km}^{ij}(\mathbf{u})u_{xx}^k u_x^m + d_{2kmn}^{ij}(\mathbf{u})u_x^k u_x^m u_x^n. \quad (18)$$

We assume that A_2 is non-degenerate, *i.e.* $\det g^{ij} \neq 0$. Operators of the above type can always be brought by a point transformation of the type $\tilde{u} = \tilde{u}(\mathbf{u})$ into the canonical form [1, 4, 27, 28]

$$A_2^{ij} = \partial_x(g^{ij}\partial_x + c_k^{ij} a_x^k)\partial_x. \tag{19}$$

Such operators have been recently studied and classified [12, 13].

Now we devote ourselves to finding the Lagrangian representation. First of all we observe that $L_n = L_n(u^k, u_x^k, u_{xx}^k, u_{xxx}^k)$. Then, since homotopy operator in the variational sequence preserves homogeneity, the functions L_n must be a homogeneous polynomials with respect to the derivative coordinates. We have another interesting property of each of the functions L_n.

Lemma 4. *The components L_n of the Lagrangian representation* (12) *are conservation law densities of the initial system* (3).

Proof. We can start the recurrence relation (3) from $\mathbf{H}_0^{(k)} = \int u^k dx$: we have $\mathbf{H}_1^{(k)} = K^{km}\int L_m dx$. Indeed, the expression $A_2^{ij}\tilde{\psi}_j$ can be expanded using formula (17); we have

$$\begin{aligned} A_2^{ij}\tilde{\psi}_j =& K^{ih}\left(\frac{\partial L_h}{\partial u_\sigma^k}\partial_\sigma\partial_x\left(K^{kj}\tilde{\psi}_j\right) + \partial_x(-1)^\sigma\partial_\sigma\left(\frac{\partial L_k}{\partial u_\sigma^h}K^{kj}\tilde{\psi}_j\right)\right) \\ =& K^{ih}\partial_x\left(\mathcal{E}(L_k)_h\right)K^{kj}\tilde{\psi}_j + \text{higher order terms,,} \end{aligned}$$

where 'higher order terms' are terms linear in $\partial_\sigma\tilde{\psi}_j$ and

$$\mathcal{E}(L_k)_h = (-1)^\sigma\partial_\sigma\left(\frac{\partial L_k}{\partial u_\sigma^h}\right)$$

is the Euler-Lagrange expression. Using the above expression in the recurrence relation yields the proof. □

The above Lemma does not help us in finding a representative of L_n of minimal order (up to total divergencies). The minimisation of order in inverse problems of the Calculus of Variations is a mathematical area which is richer in conjectures than in results, see [30] for a review on these problems. However, in concrete examples homogeneity considerations on the recurrence relation tells us that L_n can be a homogeneous polynomial of the derivative coordinates of degree not less than 2. Every homogeneous polynomial of degree 2 has the form $L_n = L^1_{nmp} u^m_x u^p_x + L^2_{nm} u^n_{xx}$, but we prefer to rewrite L_n as follows:

$$L_n = \left(\frac{1}{2} G_{nm} u^m_x + R_{nm} u^m_x\right)_x - \frac{1}{2} L_{nsm} u^s_x u^m_x. \tag{20}$$

where $G_{nm} = G_{mn}$, $R_{mn} = -R_{nm}$, $L_{nsm} = L_{nms}$. Indeed, examples show that we can actually find conservation law densities of the above form. Note that the n conservation law densities $-\frac{1}{2} L_{nsm} u^s_x u^m_x$ are not independent in general (see [14]). We need the corresponding form of A_2; it can be obtained by a straightforward computation.

Lemma 5. *If the Lagrangian representation* (L_n) *has the form* (20) *then the operator* A_2 *takes the form*

$$\begin{aligned} A_2^{ij} = {} & K^{ip} G_{pn} K^{nj} \partial_x^3 + K^{ip}(F_{pmn} + G_{pn,m} - L_{npm} - L_{pnm}) K^{nj} u^m_x \partial_x^2 \\ & + K^{ip} \left[F_{pmn} u^m_{xx} + \left(F_{pmn,s} - \frac{1}{2} L_{psm,n} - \frac{1}{2} L_{nsm,p} \right) u^s_x u^m_x \right] K^{nj} \partial_x \\ & + K^{ip} \left[L_{npm} u^m_{xx} + L_{npm,s} u^s_x u^m_x - \frac{1}{2} L_{nsm,p} u^s_x u^m_x \right]_x K^{nj}, \end{aligned} \tag{21}$$

where

$$F_{pmn} = \frac{1}{2} G_{pm,n} + \frac{1}{2} G_{np,m} - \frac{1}{2} G_{nm,p} + 2L_{npm} + R_{pm,n} + R_{mn,p} + R_{np,m}. \tag{22}$$

Now we can prove the main result of this section.

Theorem 6. *The Lagrangian representation* (12) *can be found by a Lagrangian of the form given in* (20) *if and only if the tensor*

$$T_{pmn} = F_{pmn} - \frac{1}{2} (G_{pm,n} + G_{np,m} - G_{nm,p} + 4L_{npm}) \tag{23}$$

is a closed 3*-form. In this case* G_{kp} *and* L_{smn} *can be uniquely determined while* R_{kp} *is determined up to the differential of a* 1*-form.*

Proof. If we assume that $L_n = L_n(u^k, u^k_x, u^k_{xx})$ then the expression (17) reads as

$$
\begin{aligned}
A_2^{ij} = K^{ip}\left(\frac{\partial L_p}{\partial u^n_{xx}} + \frac{\partial L_n}{\partial u^p_{xx}}\right) K^{nj}\partial_x^3 \\
+ K^{ip}\left[\frac{\partial L_p}{\partial u^n_x} - \frac{\partial L_n}{\partial u^p_x} + 3\left(\frac{\partial L_n}{\partial u^p_{xx}}\right)_x\right] K^{nj}\partial_x^2 \\
+ K^{ip}\left[\frac{\partial L_p}{\partial u^n} + \frac{\partial L_n}{\partial u^p} - 2\left(\frac{\partial L_n}{\partial u^p_x}\right)_x + 3\left(\frac{\partial L_n}{\partial u^p_{xx}}\right)_{xx}\right] K^{nj}\partial_x \\
+ K^{ip}\left[\frac{\partial L_n}{\partial u^p} - \left(\frac{\partial L_n}{\partial u^p_x}\right)_x + \left(\frac{\partial L_n}{\partial u^p_{xx}}\right)_{xx}\right]_x K^{nj}, \quad (24)
\end{aligned}
$$

Then we can plug in the above formula the expression of L_n given in (20). Keeping in mind the general expression of a homogeneous third-order operator (18) we have:

1. $G_{kn} = M_{ks} g_2^{sp} M_{pn}$ by comparing the leading coefficient, so that G_{hk} is uniquely determined;

2. $L_{smn} = M_{sk} d_{2m}^{kp} M_{pn}$ by comparing the coefficient of u^m_{xxx}, so that L_{smn} is uniquely determined;

3. $F_{pmn} = M_{ip} c_{2m}^{ij} M_{jn}$ by comparing the coefficient of u^m_{xx} in the coefficient of ∂_x, so that F_{pmn} is uniquely determined.

The equation (22) can be rewritten as

$$
R_{pm,n}+R_{mn,p}+R_{np,m} = F_{pmn}-\frac{1}{2}\left(G_{pm,n} + G_{np,m} - G_{nm,p} + 4L_{npm}\right). \quad (25)
$$

If we interpret R_{pm} as a 2-form, then the above equation is integrable as the right-hand side is a closed 3-form. As our considerations are local, we can solve the above equation (non-uniquely) by Poincaré's Lemma. □

Corollary 7. *The matrix G_{ij} in (20) is the leading term of the symplectic operator B.*

Remark 8. *In all our concrete examples (WDVV systems) the integrability requirement on the tensor T (23) is passed. We* conjecture *that this condition is a consequence of the fact that one can always obtain a minimal order potential ψ for any symplectic operator B. This conjecture goes along the lines of existing conjecture on minimal order Lagrangians for locally variational PDEs, see [30] and references therein.*

4 Lagrangian representation of WDVV equations

The main examples of bi-Hamiltonian systems of the above type that we have in mind are the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations. The WDVV system arises in $2D$ topological field theory [6, 7] as the associativity condition of an algebra in an N-dimensional space. This system is an overdetermined system of third-order PDEs in one unknown function f. The form of the equations also depend on the choice of a nondegenerate scalar product $\eta_{\alpha\beta}$ on $\mathbb{R}^N$.

The WDVV equations on an N-dimensional space can also be presented in the form of $N-2$ hydrodynamic-type systems in $N(N-1)/2$ components. Such systems are mutually commuting and non-diagonalizable (see[23] and references therein). In the first non-trivial case $N=3$ it has been showed that for three distinct choices of $\eta_{\alpha\beta}$ such systems of PDEs are bi-Hamiltonian by a pair of local operators of the type (4) and (18) (see [11, 17, 18]). The same has been proved more recently for two systems in the case $N=6$ [26].

In this section we will find the Lagrangian representation of the WDVV system from [11]. As a by-product we will find a structure formula for the third-order operator A_2 that has an independent interest.

Given a function $F=F(t^1,\dots,t^N)$ we assume that

$$\eta_{\alpha\beta}=\frac{\partial^3 F}{\partial t^1\partial t^\alpha\partial t^\beta} \tag{26}$$

is a constant nondegenerate symmetric matrix ($\eta^{\alpha\beta}$ will denote its inverse matrix); the WDVV equations are equivalent to the requirement that the functions

$$c^\alpha_{\beta\gamma}=\eta^{\alpha\mu}\frac{\partial^3 F}{\partial t^\mu\partial t^\beta\partial t^\gamma} \tag{27}$$

are the structure constants of an associative algebra. Then the associativity condition reads as

$$\eta^{\mu\lambda}\frac{\partial^3 F}{\partial t^\lambda\partial t^\alpha\partial t^\beta}\frac{\partial^3 F}{\partial t^\nu\partial t^\mu\partial t^\gamma}=\eta^{\mu\lambda}\frac{\partial^3 F}{\partial t^\nu\partial t^\alpha\partial t^\mu}\frac{\partial^3 F}{\partial t^\lambda\partial t^\beta\partial t^\gamma} \tag{28}$$

The integrability of the above equations was proved in [7] by giving a Lax pair for all values of N and $\eta_{\alpha\beta}$. The Hamiltonian geometry of WDVV equations also attracted the interest of a number of researchers [11, 20, 23]. We will focus on [11], where the case $N=3$ was considered with η antidiagonal identity, *i.e.* $\eta_{\alpha\beta}=\delta_{\alpha+\beta,4}$. In this case F is of the form

$F = \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2} t^1 (t^2)^2 + f(t^2, t^3)$, and the WDVV system consists of the single equation (after setting $x = t^2$, $t = t^3$)

$$f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xtt}. \tag{29}$$

The above equation is a third order Monge-Ampère equation (see [22] for recent results).

Let us introduce the new variables $a^1 = a = f_{xxx}$, $a^2 = b = f_{xxt}$, $a^3 = c = f_{xtt}$. Then the compatibility conditions for the WDVV equation can be written as an hydrodynamic type system of PDEs $a^i_t = v^i_j(\mathbf{a}) a^j_x$ where

$$a_t = b_x, \quad b_t = c_x, \quad c_t = (b^2 - ac)_x. \tag{30}$$

It was proved in [11] that the above system can be written as a Hamiltonian system in two ways:

$$a^i_t = A_1^{ij} \frac{\delta H_2}{\delta a^j} = A_2^{ij} \frac{\delta H_1}{\delta a^j} \tag{31}$$

with respect to two compatible local Hamiltonian operators A_1 and A_2, with expressions

$$A_1 = \begin{pmatrix} -\frac{3}{2}\partial_x & \frac{1}{2}\partial_x a & \partial_x b \\ \frac{1}{2} a \partial_x & \frac{1}{2}(\partial_x b + b\partial_x) & \frac{3}{2} c\partial_x + c_x \\ b\partial_x & \frac{3}{2}\partial_x c - c_x & (b^2 - ac)\partial_x + \partial_x (b^2 - ac) \end{pmatrix} \tag{32}$$

$$A_2 = \begin{pmatrix} 0 & 0 & \partial_x^3 \\ 0 & \partial_x^3 & -\partial_x^2 a \partial_x \\ \partial_x^3 & -\partial_x a \partial_x^2 & \partial_x^2 b \partial_x + \partial_x b \partial_x^2 + \partial_x a \partial_x a \partial_x \end{pmatrix} \tag{33}$$

and Hamiltonian densities $h_2 = c$, $h_1 = -\frac{1}{2} a (\partial_x^{-1} b)^2 - (\partial_x^{-1} b)(\partial_x^{-1} c)$, respectively. The two Hamiltonian operators A_1 and A_2 are homogeneous (see [8, 9] and the Introduction for more details).

The observation that led to finding A_1 was that in the Lax pair of the system (30)

$$\psi_x = \lambda \begin{pmatrix} 0 & 1 & 0 \\ b & a & 1 \\ c & b & 0 \end{pmatrix} \psi, \qquad \psi_t = \lambda \begin{pmatrix} 0 & 0 & 1 \\ c & b & 0 \\ b^2 - ac & c & 0 \end{pmatrix} \psi \tag{34}$$

the eigenvalues $u^k(\mathbf{a})$ of the matrix that gives the x-evolution are conservation law densities. If the system is rewritten using the above eigenvalues as new dependent variables u^k, *i.e.*, using the point transformation

$$a = u^1 + u^2 + u^3, \quad b = -\frac{1}{2}(u^1 u^2 + u^2 u^3 + u^3 u^1), \quad c = u^1 u^2 u^3 \tag{35}$$

the operator A_1 becomes evident and is of the type $A_1^{ij} = K^{ij}\partial_x$, where

$$K = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix},$$

and the Hamiltonian is $\mathcal{H}_1 = u^1u^2u^3$. In these new coordinates our system (30) takes the form

$$u^i_t = \frac{1}{2}(u^ju^k - u^iu^j - u^iu^k)_x \tag{36}$$

where i, j, k are three distinct indices.

The operator A_2 for the system (30) was found in a completely different way. More precisely, a Lagrangian for the x-derivative of the WDVV equation (29) was found, and a symplectic representation of this equation was achieved in [11]. Then A_2 was found by inverting a corresponding symplectic form and multiplying it by A_1. It is necessary to emphasize that the coordinates a, b, c (see (30)) are Casimirs for A_2, *i.e.* A_2 is in the canonical form (19) in these coordinates, and the inverse matrix of the leading term g^{ik} is the Monge metric [12]

$$g_{ij} = \begin{pmatrix} -2b & a & 1 \\ a & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{37}$$

At this point we would like to stress that *flat coordinates u^i of the first operator A_1 make the expression of the second operator A_2 much more complicated with respect to the initial coordinates a^i, since the latter are Casimirs of A_2*. On the other hand, our structure formula (21) connects the operator A_2 with conservation law densities of our system, with the leading coefficient of the symplectic operator and with a newly introduced skew-symmetric tensor R_{mn}, and all these quantities are computable in terms of flat coordinates of the first operator, thus leading to the Lagrangian representation (8). This means that we have to change coordinates to the operator A_2. To this aim we use the following formula:

$$A_2^{ij}(\mathbf{u}) = \frac{\partial u^i}{\partial a^n} A_2^{nm}(\mathbf{a}) \frac{\partial u^j}{\partial a^m}, \tag{38}$$

The leading coefficient of A_2 is the following contravariant metric $g^{ij}(\mathbf{u})$

$$\begin{aligned} g^{ii} &= \frac{3{u^i}^2 + {u^j}^2 + {u^k}^2 - 3u^iu^j - 3u^iu^k + u^ju^k}{(u^i - u^j)^2(u^i - u^k)^2} \\ &= \frac{(u^i - u^j)^2 + (u^i - u^k)^2 + (u^j - u^k)^2 + 4(u^i - u^j)(u^i - u^k)}{(u^i - u^j)^2(u^i - u^k)^2}, \end{aligned} \tag{39}$$

$$g^{ij} = \frac{-1}{(u^i - u^j)^2}, \tag{40}$$

where i, j, k are three pairwise distinct indexes. The corresponding covariant metric is

$$g_{ii} = \frac{-3(u^j - u^k)^2}{4}, \tag{41}$$

$$\begin{aligned} g_{ij} &= \frac{1}{4}(-2{u^i}^2 - 2{u^j}^2 - 3{u^k}^2 + u^iu^j + 3u^iu^k + 3u^ju^k) \\ &= -\frac{1}{4}((u^i - u^j)^2 + (u^i - u^k)^2 + (u^j - u^k)^2 + (u^i - u^j)(u^i - u^k)), \end{aligned} \tag{42}$$

where i, j, k are three pairwise distinct indexes.

The leading term $G_{ij}(\mathbf{u})$ of the symplectic operator is

$$\begin{aligned} G_{ii} &= \frac{(u^i)^2 + (u^j)^2 + (u^k)^2 - u^iu^j - u^iu^k - u^ju^k}{(u^i - u^j)^2(u^i - u^k)^2} \\ &= \frac{(u^i - u^j)^2 + (u^i - u^k)^2 + (u^j - u^k)^2}{2(u^i - u^j)^2(u^i - u^k)^2}, \end{aligned} \tag{43}$$

$$\begin{aligned} G_{ij} &= \frac{(u^i)^2 + (u^j)^2 - (u^k)^2 + u^iu^k + u^ju^k - 3u^iu^j}{(u^i - u^j)^2(u^i - u^k)(u^j - u^k)} \\ &= \frac{(u^i - u^j)^2 + (u^i - u^k)^2 + (u^j - u^k)^2 - 4(u^k - u^i)(u^k - u^j)}{2(u^i - u^j)^2(u^i - u^k)(u^j - u^k)}, \end{aligned} \tag{44}$$

where i, j, k are a triplet of distinct indices. Note that

$$\det \mathbf{G} = -\frac{16}{(u^1 - u^2)^2(u^1 - u^3)^2(u^2 - u^3)^2}, \tag{45}$$

and from $G_{11} > 0$ (outside obvious singularities) it follows that the signature of the metric is $(2, 1)$. It is also remarkable that the metric G_{ij} has constant sectional curvature $-1/16$. The corresponding contravariant metric is

$$\begin{aligned} G^{ii} &= -\frac{1}{4}(u^i - u^j)(u^i - u^k), \\ G^{ij} &= -\frac{1}{4}(u^i - u^j)^2. \end{aligned}$$

The expressions of $L_{ijk} = L_{ikj}$ are:

1. when $j \neq 1$ and $k \neq 1$ or when $j = k = 1$

$$L_{1jk} = \frac{((u^1 - u^2) + (u^1 - u^3))(u^a - u^b)(u^c - u^d)}{2(u^1 - u^2)^3(u^1 - u^3)^3} \qquad (46)$$

   where (a, j, b), (c, d, k) are triplets of distinct indices with $a < b$, $c < d$;

2. when $(1, j, k)$ are a triplet of distinct indices.

$$L_{11k} = -\frac{(u^1 - u^k)^2 + (u^1 - u^j)^2}{2(u^1 - u^k)^2(u^1 - u^j)^3} \qquad (47)$$

3. when $i \neq 1$ the expressions of L_{ijk} are obtained by a cyclic permutation of the above expressions.

Remark 9. *The conserved quantities $\mathcal{L}_n = -\frac{1}{2}L_{nmp}u^m_x u^p_x$ coincide with the conserved quantities $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ defined in the paper [11]: $\mathcal{L}_n = \mathcal{I}_n$.*

Remark 10. *There is a further conservation law density in the form $u^i L_i$; however, this density is trivial as it is in the form of a total divergence.*

The integrability condition is fulfilled in our case as T_{pmn} is a completely skew-symmetric tensor, and by dimensional reasons. The single equation for R_{mn} is

$$\left(\frac{\partial R_{31}}{\partial u^2} + \frac{\partial R_{23}}{\partial u^1} + \frac{\partial R_{12}}{\partial u^3}\right)(u^1 - u^2)(u^1 - u^3)(u^2 - u^3) + 1 = 0.$$

in the unknown functions R_{23}, $R_{13} = -R_{31}$, R_{12} of the coordinates u^1, u^2, u^3. This equation can be regarded as an equation of the form $dR = -1/((u^1 - u^2)(u^1 - u^3)(u^2 - u^3))du^1 \wedge du^2 \wedge du^3$, where $R = R_{12}du^1 \wedge du^2 + R_{13}du^1 \wedge du^3 + R_{23}du^2 \wedge du^3$. One distinguished solution can be obtained by letting $\partial R_{31}/\partial u^2 = \partial R_{23}/\partial u^1 = \partial R_{12}/\partial u^3$. In this case we have

$$R^0_{ij} = -\frac{1}{3}\left(\frac{1}{(u^i - u^j)(u^i - u^k)} - \frac{1}{(u^j - u^i)(u^j - u^k)}\right) \qquad (48)$$

where i, j, k are distinct indices.

It is possible to cross-check the above results by a direct calculation of the quantity $L_\tau A_1$ aimed at verifying that $L_\tau A_1 = A_2$. This has been done using the correspondence between multivectors and superfunctions in [15, 16] and the corresponding Schouten bracket formula from [19]. The result was confirmed after 18 hours of computation with CDE, using 7.7GB of RAM, on the workstation `sophus2` of the Department of Mathematics and Physics of the Università del Salento.

5 Conclusions

A bi-Hamiltonian structure for WDVV equation with different $\eta_{\alpha\beta}$ in the case $N = 3$ also has been considered. For instance, in [17] a different identification of the variables t^2 and t^3 as t and x leads to different WDVV equations and different bi-Hamiltonian formulations through local Dubrovin–Novikov operators. Moreover, in [18] another choice of constants in $\eta_{\alpha\beta}$ was investigated. Its third order operator of [18] lies in a different class with respect to the third order operator of [11], with respect to the classification in [12]. Since in the above cases different choices of η can be connected by transformations of independent variables to the choice of the WDVV equation that we considered (29), we expect that the Lagrangian representation can be effectively achieved by the techniques exposed in this paper.

We stress that Hamiltonian operators for the $N = 3$ WDVV equation (29) in the original unknown f have been found in [20]. However, they are nonlocal and depend explicitly on independent variables and it is not clear (and a nontrivial problem) how to relate them to the operator in [11].

When $N = 4$ in (26) and choosing $\eta_{\alpha\beta}$ to be the antidiagonal identity we obtain a system of equations in third-order derivatives of f that can be rewritten as two commuting hydrodynamic-type systems in 6 unknown functions. These systems have been recently proved to be bi-Hamiltonian [26], and a Lagrangian representation have been computed with the above methods. However, its expression is more complicated and less 'regular' than the above expressions for $N = 3$. For this reason we did not decide to write down the expressions, even if we will make them available upon request. Here the dimension of the problem does not automatically imply the integrability condition $dT = 0$, and the fact that this condition holds is remarkable.

We stress that our methods might be carried out for an arbitrary homogeneous operator A_2 with no conceptual differences with the third-order case.

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References

[1] *A.V. Balandin, G.V. Potemin*, On non-degenerate differential-geometric Poisson brackets of third order, Russian Mathematical Surveys **56** No. 5 (2001) 976-977.

[2] *A. V. Bocharov, V. N. Chetverikov, S. V. Duzhin, N. G. Khor'kova, I. S. Krasil'shchik, A. V. Samokhin, Yu. N. Torkhov, A. M. Verbovetsky and A. M. Vinogradov*: Symmetries and Conservation Laws for Differential Equations of Mathematical Physics, I. S. Krasil'shchik and A. M. Vinogradov eds., Translations of Math. Monographs **182**, Amer. Math. Soc. (1999).

[3] *L. Degiovanni, F. Magri, V. Sciacca*, On deformation of Poisson manifolds of hydrodynamic type, Comm. Math. Phys. 253 (2005), 1–24.

[4] *P.W. Doyle*, Differential geometric Poisson bivectors in one space variable, J. Math. Phys. **34** No. 4 (1993) 1314-1338.

[5] *I. Dorfman*, Dirac structures and integrability of nonlinear evolution equations, John Wiley & Sons, England, 1993.

[6] *B.A. Dubrovin,* Integrable systems in topological field theory, Nucl. Phys. B, **379** (1992) 627–689.

[7] *B.A. Dubrovin*, Geometry of 2D topological field theories, Lecture Notes in Math. 1620, Springer-Verlag (1996) 120–348.

[8] *B.A. Dubrovin and S.P. Novikov*, Hamiltonian formalism of one-dimensional systems of hydrodynamic type and the Bogolyubov-Whitham averaging method, Soviet Math. Dokl. **27** No. 3 (1983) 665–669.

[9] *B.A. Dubrovin and S.P. Novikov*, Poisson brackets of hydrodynamic type, Soviet Math. Dokl. **30** No. 3 (1984), 651–2654.

[10] *B.A. Dubrovin and Y. Zhang*, Normal forms of integrable PDEs, Frobenius manifolds and Gromov-Witten invariants, math.DG/0108160.

[11] *E.V. Ferapontov, C.A.P. Galvao, O. Mokhov, Y. Nutku*, Bi-Hamiltonian structure of equations of associativity in 2-d topological field theory, Comm. Math. Phys. **186** (1997) 649-669.

[12] *E.V. Ferapontov, M.V. Pavlov, R.F. Vitolo*, Projective-geometric aspects of homogeneous third-order Hamiltonian operators, J. Geom. Phys. 85 (2014), 16–28.

[13] *E.V. Ferapontov, M. V. Pavlov, R.F. Vitolo*, Towards the classification of homogeneous third-order Hamiltonian operators, to appear in Int. Math. Res. Notices 2016. ArXiv: http://arxiv.org/abs/1508.02752.

[14] *E.V. Ferapontov, R. Sharipov*, On first-order conservation laws for systems of hydrodynamic type equations, Theoretical and Mathematical Physics, Vol. 108, No. I, 1996, 937–952.

[15] *E. Getzler*, A Darboux theorem for Hamiltonian operators in the formal calculus of variations, Duke J. Math. **111** (2002), 535-560.

[16] *S. Igonin, A. Verbovetsky, R. Vitolo,* Variational multivectors and brackets in the geometry of jet spaces. In *Symmetry in Nonlinear Mathematical Physics. Part 3*, pages 1335–1342. Institute of Mathematics of NAS of Ukraine, Kiev, 2003.

[17] *J. Kalayci, Y. Nutku*, Bi-Hamiltonian structure of a WDVV equation in 2d topological field theory, Phys. Lett. A **227** (1997), 177–182.

[18] *J. Kalayci, Y. Nutku*, Alternative bi-Hamiltonian structures for WDVV equations of associativity, J. Phys. A: Math. Gen. **31** (1998) 723-734.

[19] *P. Kersten, I. Krasil'shchik, A. Verbovetsky,* Hamiltonian operators and ℓ^*-coverings. J. Geom. Phys., **50** (2004), 273–302.

[20] *P. Kersten, I. Krasil'shchik, A. Verbovetsky, R. Vitolo*, On integrable structures for a generalized Monge-Ampere equation , Theor. Math. Phys. **128**, no. 2 (2012), 600–615.

[21] *P. Lorenzoni*, Deformations of bihamiltonian structures of hydrodynamic type, J. Geom. Phys. **44** no. 2-3 (2002), 331–375; https://arxiv.org/abs/nlin/0108015

[22] *G. Manno, G. Moreno*, Meta-Symplectic Geometry of 3rd Order Monge–Ampère Equations and their Characteristics, SIGMA **12** (2016), 032 (35 pages); http://arxiv.org/abs/1403.3521

[23] *O.I. Mokhov*, Symplectic and Poisson structures on loop spaces of smooth manifolds, and integrable systems, Russian Math. Surveys **53** No. 3 (1998) 515-622.

[24] *Y. Nutku, M.V. Pavlov*, Multi Lagrangians for Integrable Systems, Journal of Mathematical Physics 43 (3), 1441 (2002); http://dx.doi.org/10.1063/1.1427765.

[25] *P. Olver*, Applications of Lie Groups to Differential Equations, Springer GTM 107, 1993.

[26] *M.V. Pavlov, R.F. Vitolo*, On the bi-Hamiltonian Geometry of WDVV Equations, Lett. Math. Phys. **105** no. 8 (2015), 1135–1163. http://arxiv.org/abs/1409.7647.

[27] *G.V. Potemin*, On third-order Poisson brackets of differential geometry, Russ. Math. Surv. **52** (1997) 617-618.

[28] *G.V. Potemin*, Some aspects of differential geometry and algebraic geometry in the theory of solitons. PhD Thesis, Moscow, Moscow State University (1991) 99 pages.

[29] *A. Sergeev*, A Simple Way of Making a Hamiltonian System Into a Bi-Hamiltonian One, Acta Applicandae Mathematicae 83: 183–197, 2004.

[30] *R. Vitolo*, Variational sequences, in 'Handbook of Global Analysis', ed. D. Krupka, D. Saunders, Elsevier 2007.

[31] *R. Vitolo*, CDE: a Reduce package for computations in the goemetry of differential equations, software, user guide and examples freely available at http://gdeq.org (Version 1.0, October 2014). See also the Reduce website: http://reduce-algebra.sourceforge.net/