On the bi-Hamiltonian Geometry of WDVV Equations

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Abstract

We consider the WDVV associativity equations in the four dimensional case. These nonlinear equations of third order can be written as a pair of six component commuting two-dimensional non-diagonalizable hydrodynamic type systems. We prove that these systems possess a compatible pair of local homogeneous Hamiltonian structures of Dubrovin–Novikov type (of first and third order, respectively).

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1 Introduction

The Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) associativity equations arise as the conditions of associativity of an algebra in an N dimensional space. The mathematical theory of the equations can be found in [6], as well as references to papers with the physical motivations. The equations are a system of third-order PDEs in one unknown function $F = F(t^1, \dots, t^N)$. Namely, it is assumed that

$$\eta_{\alpha\beta} = \frac{\partial^3 F}{\partial t^1 \partial t^\alpha \partial t^\beta}$$

is a constant nondegenerate symmetric matrix ($\eta^{\alpha\beta}$ will denote its inverse matrix); the WDVV associativity equations are equivalent to the requirement that the functions

$$c^{\alpha}_{\beta\gamma} = \eta^{\alpha\mu}\frac{\partial^3 F}{\partial t^\mu \partial t^\beta \partial t^\gamma}$$

are the structure constants of an associative algebra. Then the associativity condition reads as

$$\eta^{\mu\lambda}\frac{\partial^3 F}{\partial t^\lambda \partial t^\alpha \partial t^\beta}\frac{\partial^3 F}{\partial t^\nu \partial t^\mu \partial t^\gamma} = \eta^{\mu\lambda}\frac{\partial^3 F}{\partial t^\nu \partial t^\alpha \partial t^\mu}\frac{\partial^3 F}{\partial t^\lambda \partial t^\beta \partial t^\gamma}$$

The integrability of the above equations was proved in [6] by giving a Lax pair for all values of N and $\eta_{\alpha\beta}$. The Hamiltonian geometry of WDVV associativity equations also attracted interest of a number of researchers. In particular, a fundamental contribution was given in papers [11, 10], where the case $N = 3$ was considered. When $N = 3$ we have the first nontrivial case with just one WDVV associativity equation. If the matrix $\boldsymbol{\eta}$ is antidiagonal, *i.e.* $\eta_{\alpha\beta} = \delta_{\alpha+\beta,4}$ and $F = \frac{1}{2}(t^1)^2t^3 + \frac{1}{2}t^1(t^2)^2 + f(t^2, t^3)$, the WDVV associativity equation is (after setting $x = t^2$, $t = t^3$)

$$f_{ttt} = f_{xxt}^2 - f_{xxx}f_{xtt}. \tag{1}$$

Introducing the new variables $a^1 = a = f_{xxx}$, $a^2 = b = f_{xxt}$, $a^3 = c = f_{xtt}$, the WDVV associativity equation can be written as an hydrodynamic type system of PDEs

$$a^i_t = v^i_j(\mathbf{a})a^j_x; \tag{2}$$

more precisely, $a_t = b_x$, $b_t = c_x$, $c_t = (b^2 - ac)_x$. It was proved in [10] that the above system can be rewritten as a bi-Hamiltonian system

$$a^i_t = A_1^{ij}\frac{\delta H_2}{\delta a^j} = A_2^{ij}\frac{\delta H_1}{\delta a^j}. \tag{3}$$

with respect to two compatible local Hamiltonian operators $\hat{A}_1$ and $\hat{A}_2$, with expressions

$$\hat{A}_1 = \begin{pmatrix} -\frac{3}{2}\partial_x & \frac{1}{2}\partial_x a & \partial_x b \\ \frac{1}{2}a\partial_x & \frac{1}{2}(\partial_x b + b\partial_x) & \frac{3}{2}c\partial_x + c_x \\ b\partial_x & \frac{3}{2}\partial_x c - c_x & (b^2 - ac)\partial_x + \partial_x(b^2 - ac) \end{pmatrix}$$

$$\hat{A}_2 = \begin{pmatrix} 0 & 0 & \partial_x^3 \\ 0 & \partial_x^3 & -\partial_x^2 a\partial_x \\ \partial_x^3 & -\partial_x a\partial_x^2 & \partial_x^2 b\partial_x + \partial_x b\partial_x^2 + \partial_x a\partial_x a\partial_x \end{pmatrix}$$

and Hamiltonian densities, respectively, $h_2 = c$, $h_1 = -\frac{1}{2}a(\partial_x^{-1}b)^2 - (\partial_x^{-1}b)(\partial_x^{-1}c)$, where $H_i = \int h_i dx$. Here by a 'Hamiltonian operator' we mean an operator $\hat{A}$ such that its Schouten bracket $[\hat{A}, \hat{A}]$ vanishes, and by 'compatible' (or commuting) operators we mean that the Schouten bracket $[\hat{A}_1, \hat{A}_2]$ vanishes (see [4] for the definition).

The two Hamiltonian operators $\hat{A}_1$ and $\hat{A}_2$ are members of a class of operators which has been introduced by Dubrovin and Novikov [7, 8]. They are homogeneous with respect to the grading $\deg a^i = 0$, $\deg \partial_x = 1$. They have interesting geometric properties which were completely described in [7] for first order Hamiltonian operators. Third order Hamiltonian operators have a more complicated structure (see [24, 25, 26, 1, 5]). A complete classification of these Hamiltonian operators was found just in two and three component cases (see detail in [13]).

In particular, $\hat{A}_1$ was found in [11], and it is completely specified by a contravariant flat pseudo-Riemannian metric g^{ik}. The observation that led to finding $\hat{A}_1$ was that the eigenvalues $u^k(\mathbf{a})$ of one of the matrices of the Lax pair of the system (2) are conservation law densities. If the system is rewritten using the above eigenvalues as new dependent variables u^k, the Hamiltonian operator $\hat{A}_1$ becomes evident and is of the type $A_1^{ij} = K^{ij}\partial_x$, where K^{ij} is a constant symmetric non-degenerate matrix. Hamiltonian operators of this type are said to be *hydrodynamic type Hamiltonian operators*.

The Hamiltonian operator $\hat{A}_2$ was found in a completely different way. More precisely, a Lagrangian for the x-derivative of the WDVV associativity equation (1) was found, and a symplectic representation of this equation was achieved in [10]. Then $\hat{A}_2$ was found by inverting the corresponding symplectic form and multiplying it by $\hat{A}_1$. It is necessary to emphasize that the coordinates a^k (see (2)) are Casimirs densities for $\hat{A}_2$, *i.e.* $\hat{A}_2$ has vanishing free term in these coordinates. It is known [24, 25, 26] that $\hat{A}_2$ has a particularly simple form (16) when written with respect to its Casimirs densities; in these coordinates a^k the inverse matrix of the leading coefficient g^{ik} is a Monge metric [13], which in this example reads as

$$g_{ij} = \begin{pmatrix} -2b & a & 1 \\ a & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Other approaches to the Hamiltonian geometry of WDVV equations appeared in the literature in the case $N = 3$. For instance, in [16] a different identification of the variables t^2 and t^3 as t and x leads to different WDVV associativity equations and different bi-Hamiltonian formulations through local Dubrovin–Novikov operators of the first and third order. Moreover, in [17] another choice of constants in $\eta_{\alpha\beta}$ was investigated. Its third order Hamiltonian operator lies in a different class (cf. [10]) with respect to the classification in [13]. Finally, in [19] Hamiltonian operators for the WDVV equation in the non-evolutionary form (1) have been found (although with an explicit dependency on the independent variables).

In the case $N = 4$ the bi-Hamiltonian nature of WDVV associativity equations was an open problem. If the matrix $\boldsymbol{\eta}$ is antidiagonal[1], *i.e.* the choice $\eta_{\alpha\beta} = \delta_{\alpha+\beta,5}$, WDVV associativity equations were considered in [6]. This implies $F = \frac{1}{2}(t^1)^2t^4 + t^1t^2t^3 + f(t^2, t^3, t^4)$. Then we have the following WDVV associativity equations (after the identifications $x = t^2$, $y = t^3$, $z = t^4$)

$$\begin{aligned}
&-2f_{xyz} - f_{xyy}f_{xxy} + f_{yyy}f_{xxx} = 0,\\
&-f_{xzz} - f_{xyy}f_{xxz} + f_{yyz}f_{xxx} = 0,\\
&-2f_{xyz}f_{xxz} + f_{xzz}f_{xxy} + f_{yzz}f_{xxx} = 0,\\
&-f_{yyy}f_{xxz} + f_{yzz} + f_{yyz}f_{xxy} = 0,\\
&f_{zzz} - (f_{xyz})^2 + f_{xzz}f_{xyy} - f_{yyz}f_{xxz} + f_{yzz}f_{xxy} = 0,\\
&f_{yyy}f_{xzz} - 2f_{yyz}f_{xyz} + f_{yzz}f_{xyy} = 0.
\end{aligned} \tag{4}$$

In [12] (see also [21]) it was found that the above system can be rewritten as a compatible pair of two hydrodynamic type systems (see Section 2). Such systems admit a Dubrovin–Novikov type first-order Hamiltonian operator [12], but a second Dubrovin–Novikov type third order Hamiltonian operator for these systems was never found until now (see basic facts in 2.1).

The main task of this paper is to completely uncover the bi-Hamiltonian geometry of the WDVV associativity equations for $N = 4$ and the antidiagonal metric $\eta_{\alpha\beta} = \delta_{\alpha+\beta,5}$ by finding a second Dubrovin–Novikov type third order Hamiltonian operator for the corresponding hydrodynamic-type systems. In order to achieve this goal, we develop a procedure that can be fruitfully used to find such Hamiltonian operators for other WDVV associativity systems, for example for different matrices $\boldsymbol{\eta}$ or different values of N.

Below, we outline our procedure for convenience of the reader.

[1]The constant symmetric matrix $\boldsymbol{\eta}$ can always be reduced by a linear change of the coordinates (t^i) to either the antidiagonal form if $\eta_{11} = 0$, or to another form if $\eta_{11} \neq 0$. Only the first case admits physically relevant examples. See [6] for details.

1. We start from the following data: the Lax pair and the first-order Hamiltonian structure $\hat{A}_1$.
2. In flat coordinates $(\mathbf{u})$ of $\hat{A}_1$, using the Lax pair, it is possible to find a sequence of homogeneous conservation law densities in the most compact form.
3. If we suppose that the above densities are related by a bi-Hamiltonian recursion with an unknown third-order operator $\hat{A}_2$ of Dubrovin–Novikov type, we can find a candidate to be the metric $g_{ij}(\mathbf{u})$ that is the coefficient of ∂_x^3 (this is the content of Theorem 3).
4. We make the hypothesis that the Casimirs of $\hat{A}_2$ are the original coordinates $(\mathbf{a})$ in which the hydrodynamic-type WDVV systems are written. Indeed, this is the case: if we change coordinates from $(\mathbf{u})$ to $(\mathbf{a})$ we can prove that $g_{ij}(\mathbf{a})$ is a Monge metric and completely determines a Hamiltonian operator $\hat{A}_2$ which is compatible with $\hat{A}_1$, according with the general theory in [13] (Theorem 15).
5. In the same Theorem we also exhibit a factorization of the Hamiltonian operator whose theoretical existence was ensured in [5].
6. A deeper analysis of the properties of the factorization of the third-order Hamiltonian operator of Dubrovin–Novikov type performed in Section 3 applied to the WDVV hydrodynamic type systems allows us to formulate general existence theorems for nonlocal Casimirs of the Dubrovin–Novikov type third-order Hamiltonian operator, Hamiltonians for the WDVV hydrodynamic type systems and momentum, thus completing a bi-Hamiltonian picture of the WDVV hydrodynamic type systems in six components.

The above procedure can be fruitfully used not only for WDVV associativity equations but for any system of PDEs in $1+1$ dimensions.

We stress that the reconstruction of the Dubrovin–Novikov type third-order Hamiltonian operator for the six component systems involves several steps where coordinate expressions of the objects involved can only be handled by computer algebra systems, and too big to be written down in this paper, However, the final result, thanks to the factorized form that we achieve in Theorem 15, is so compact that Hamiltonian properties of the WDVV hydrodynamic type systems can be checked by pen and paper.

In particular, almost all symbolic computations were performed by CDE [27], a REDUCE package for integrability of PDEs. Only the linearization of the WDVV hydrodynamic type systems and their formal adjoint have been done with Jets [2], a Maple package for the geometry of PDEs, since this feature will only appear in the forthcoming version of CDE. The computation of homogeneous conservation law densities required about 6 hours and 18GB of RAM on the server `sophus` of the Dipartimento di Matematica e Fisica “E. De Giorgi” of the Università del Salento.

We stress that despite the fact that we heavily relied on computer algebra calculations, our main results are checkable by pen and paper, in particular the Hamiltonian property (with respect to the newly found third order operator $\hat{A}_2$) of the 6-component WDVV hydrodynamic type systems; see Remark 17.

At the end of the paper a conclusive section contains several interesting remarks about the perspectives of mathematical research on the bi-Hamiltonian geometry of WDVV associativity equations. Here we would like to stress that at this point it is natural to conjecture that every WDVV system admits a third-order local Hamiltonian operator of Dubrovin–Novikov type, with a distinguished subset of WDVV systems admitting a bi-Hamiltonian formulation. Note that a Dubrovin–Novikov type third order Hamiltonian operator is completely specified by an object from projective geometry, a quadratic line complex [3, 13], which could be of interest in the rich geometric framework that surrounds WDVV associativity equations.

2 The WDVV Associativity Equations in 4 Dimensions

In this section we will give a brief summary of what is known from the previous investigations in [12] (see also [21]).

For general N the WDVV associativity equations can be presented as $N-2$ commuting two-dimensional non-diagonalizable hydrodynamic type systems with $n = N(N-1)/2$ components. The procedure for obtaining such systems is shown here in details in the case $N = 4$ (4). We call them WDVV hydrodynamic type systems here and below.

We introduce new field variables a^k in correspondence with every derivative $f_{t^i t^j t^k}$ which contains at least one instance of $x = t^2$, i.e. $a^1 = f_{xxx}, a^2 = f_{xxy}, a^3 = f_{xxz}, a^4 = f_{xyy}, a^5 = f_{xyz}, a^6 = f_{xzz}$. Then, the compatibility conditions for the WDVV associativity equations (4) can be written as a pair of hydrodynamic type systems (cf. (2))

$$a^i_y = v^i_j(\mathbf{a})a^j_x, \quad a^i_z = w^i_j(\mathbf{a})a^j_x;$$

more precisely

$$a^i_y = (v^i(\mathbf{a}))_x, \quad a^i_z = (w^i(\mathbf{a}))_x, \tag{5}$$

where

$$v^1 = a^2, \quad w^1 = a^3, \quad v^2 = a^4, \quad v^3 = w^2 = a^5, \quad w^3 = a^6, \quad v^4 = f_{yyy} = \frac{2a^5 + a^2a^4}{a^1}, \quad v^5 = w^4 = f_{yyz} = \frac{a^3a^4 + a^6}{a^1},$$

$$v^6 = w^5 = f_{yzz} = \frac{2a^3a^5 - a^2a^6}{a^1}, \quad w^6 = f_{zzz} = (a^5)^2 - a^4a^6 + \frac{(a^3)^2a^4 + a^3a^6 - 2a^2a^3a^5 + (a^2)^2a^6}{a^1}.$$

The commuting hydrodynamic type systems (5) can be expressed as the compatibility condition of two Lax pairs with a common part [12]. Such a common part takes the form

$$\begin{pmatrix}\psi\\ \psi_1\\ \psi_2\\ \psi_3\end{pmatrix}_x = \lambda\mathbf{A}\begin{pmatrix}\psi\\ \psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = \lambda\begin{pmatrix}0 & 1 & 0 & 0\\ a^3 & a^2 & a^1 & 0\\ a^5 & a^4 & a^2 & 1\\ a^6 & a^5 & a^3 & 0\end{pmatrix}\begin{pmatrix}\psi\\ \psi_1\\ \psi_2\\ \psi_3\end{pmatrix}. \tag{6}$$

The characteristic equation of $\mathbf{A}$ is

$$\det(\mathbf{A} - \rho\mathbf{I}) = \rho^4 - 2a^2\rho^3 + [(a^2)^2 - 2a^3 - a^1a^4]\rho^2 + 2(a^2a^3 - a^1a^5)\rho + (a^3)^2 - a^1a^6 = 0. \tag{7}$$

It can be proved that its roots u^1, u^2, u^3, u^4 are conservation law densities of both systems (5). The relation between these roots and the coefficients of the characteristic equation is given by the Viète formulae

$$\begin{aligned}
a^2 &= \frac{1}{2}(u^1 + u^2 + u^3 + u^4),\\
a^3 &= \frac{1}{4}[(u^1)^2 + (u^2)^2 + (u^3)^2 + (u^4)^2] - \frac{1}{8}(u^1 + u^2 + u^3 + u^4)^2 - \frac{1}{2}a^1a^4,\\
a^5 &= \frac{1}{2a^1}(2a^2a^3 + u^1u^2u^3 + u^1u^2u^4 + u^1u^3u^4 + u^2u^3u^4),\\
a^6 &= \frac{1}{a^1}[(a^3)^2 - u^1u^2u^3u^4].
\end{aligned} \tag{8}$$

We can change the variables a^k of the systems (5) to the new variables $u(\mathbf{a})$, i.e. $u^0 = a^1$, u^1, u^2, u^3, u^4, $u^5 = a^4$. These conservation law densities are flat coordinates (see details in [12]), i.e. hydrodynamic type systems (5) can be equipped by the Hamiltonian operator $\hat{A}_1 = \mathbf{K}\partial_x$, where $\mathbf{K}$ is the constant symmetric nondegenerate matrix

$$K^{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & -1 & 1 & -1 & -1 & 0 \\ 0 & -1 & -1 & 1 & -1 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{9}$$

namely

$$u^i_y = K^{ij}\partial_x \frac{\delta \mathbf{H}_7}{\delta u^j}, \quad u^i_z = K^{ij}\partial_x \frac{\delta \mathbf{H}_8}{\delta u^j}, \tag{10}$$

where the Hamiltonian densities $h_7 = a^5$ and $h_8 = \frac{1}{2}a^6$, while the momentum density $h_6 = a^3$.

In the original coordinates $a^k(\mathbf{u})$ this Dubrovin–Novikov type first order Hamiltonian operator becomes

$$A_1^{ij} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ a^1 & a^2 & a^3 & a^4 & a^5 & a^6 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ a^2 & a^4 & a^5 & R & P & S \\ 2a^3 & 2a^5 & 2a^6 & 2P & 2S & 2Q \end{pmatrix}\partial_x + \partial_x \begin{pmatrix} 0 & 0 & a^1 & -1 & a^2 & 2a^3 \\ 0 & -1 & a^2 & 0 & a^4 & 2a^5 \\ 0 & 0 & a^3 & 0 & a^5 & 2a^6 \\ -1 & 0 & a^4 & 0 & R & 2P \\ 0 & 0 & a^5 & 0 & P & 2S \\ 0 & 0 & a^6 & 0 & S & 2Q \end{pmatrix}, \tag{11}$$

where

$$P = \frac{a^3a^4 + a^6}{a^1}, \quad R = \frac{2a^5 + a^2a^4}{a^1}, \quad S = \frac{2a^3a^5 - a^2a^6}{a^1}, \quad Q = (a^5)^2 - a^4a^6 + \frac{(a^3)^2a^4 + a^3a^6 - 2a^2a^3a^5 + (a^2)^2a^6}{a^1}.$$

2.1 The Structure of the Second Hamiltonian Operator

Having in mind the fundamental examples of the case $n = N = 3$ [10, 16, 17] we *conjecture* that also our six-component systems admit a Dubrovin–Novikov type third-order Hamiltonian operator, denoted by $\hat{A}_2$.

Any bi-Hamiltonian hierarchy

$$u^i_{t^k} = A_2^{is}\frac{\delta \mathbf{H}_k}{\delta u^s} = A_1^{is}\frac{\delta \mathbf{H}_{k+1}}{\delta u^s}, \quad i = 1, ..., n \tag{12}$$

is determined by two compatible Hamiltonian operators $\hat{A}_1$ and $\hat{A}_2$, if their Schouten bracket vanishes: $[\hat{A}_1, \hat{A}_2] = 0$. We recall that the Hamiltonian property of a differential operator $\hat{A}$ is equivalent to its formal skew-adjointness $\hat{A}^* = -\hat{A}$ and $[\hat{A}, \hat{A}] = 0$, or the fact that $\hat{A}$ defines a Poisson bracket on the space of conservation law densities:

$$\{\mathbf{H}_k, \mathbf{H}_m\}_{\hat{A}} = \int \frac{\delta \mathbf{H}_k}{\delta u^i} A^{ij} \frac{\delta \mathbf{H}_m}{\delta u^j} dx$$

According to Magri's Theorem (see [20]) all functionals $\mathbf{H}_p = \int h_p(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, ...)dx$ where h_p is a conservation law density commute with each other under either one of the Poisson brackets defined by the operators (i.e. $\{\mathbf{H}_k, \mathbf{H}_m\}_1 = 0$ and independently $\{\mathbf{H}_k, \mathbf{H}_m\}_2 = 0$). In general, the reconstruction of a bi-Hamiltonian nature of an integrable evolutionary system starting from the hierarchy of conservation law densities is a very complicated problem even if both Hamiltonian operators are local. Now we assume that a given evolutionary system

$$u^i_t = U^i(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, ...)$$

has a compatible pair of Dubrovin–Novikov type first and third order Hamiltonian operators $\hat{A}_1$, $\hat{A}_2$, respectively. This means that the above evolutionary system can be written in two different forms (see (12))

$$u^i_t = A_2^{is}\frac{\delta \mathbf{H}_1}{\delta u^s} = A_1^{is}\frac{\delta \mathbf{H}_2}{\delta u^s}, \quad i = 1, ..., n, \tag{13}$$

where (see detail in [7] and [8])

$$A_1^{ij} = g_1^{ij}(\mathbf{u})\partial_x + b_{1k}^{ij}(\mathbf{u})u_x^k, \tag{14}$$

$$A_2^{ij} = g_2^{ij}(\mathbf{u})\partial_x^3 + b_{2k}^{ij}(\mathbf{u})u_x^k\partial_x^2 + [c_{2k}^{ij}(\mathbf{u})u_{xx}^k + c_{2km}^{ij}(\mathbf{u})u_x^k u_x^m]\partial_x \\ + d_{2k}^{ij}(\mathbf{u})u_{xxx}^k + d_{2km}^{ij}(\mathbf{u})u_{xx}^k u_x^m + d_{2kmp}^{ij}(\mathbf{u})u_x^k u_x^m u_x^p, \tag{15}$$

while the Hamiltonian functionals $\mathbf{H}_1 = \int h_1(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, ...)dx$ and $\mathbf{H}_2 = \int h_2(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, ...)dx$ will be specified in the next Subsection.

Let us recall some basic facts on the geometry of homogeneous first and third-order Hamiltonian operators. We suppose that these Hamiltonian operators have non-degenerate leading terms, i.e. $\det g_1^{ij} \neq 0$ and $\det g_2^{ij} \neq 0$. The Hamiltonian operators (14) and (15) are form-invariant under point transformations of the dependent variables, $\tilde{u} = \tilde{u}(\mathbf{u})$. In this case the coefficients of both Hamiltonian operators (14) and (15) transform as differential-geometric objects. For instance, g_1^{ij} and g_2^{ij} transform as a $(2,0)$-tensors, so that their inverse g_{ij}^1 and g_{ij}^2 define pseudo-Riemannian metrics (note that the second one is not flat in general), the expressions $g_{js}^1 b_{1k}^{si}$, $-\frac{1}{3}g_{js}^2 b_{2k}^{si}$, $-\frac{1}{3}g_{js}^2 c_{2k}^{si}$, $-g_{js}^2 d_{2k}^{si}$ transform as Christoffel symbols of affine connections, etc. [8]. B.A. Dubrovin and S.P. Novikov proved that the $\hat{A}_1$ is Hamiltonian if and only if $\Gamma_{1jk}^i = -g_{js}^1 b_{1k}^{si}$ is the Levi-Civita connection of the metric g_{ij}^1 and is flat [7]. It was conjectured by S.P. Novikov that the last connection, $\Gamma_{2jk}^i = -g_{js}^2 d_{2k}^{si}$ must be symmetric (with respect to low indices) and flat; this was confirmed in [26], see also [5]. Therefore, there exists a coordinate system $a^k(\mathbf{u})$ such that Γ_{2jk}^i vanish. These coordinates are determined up to affine transformations. We note here that a^i are nothing but the conservation law densities of Casimirs of $\hat{A}_2$. In these coordinates a^k the last three terms in (15) vanish, leading to the simplified expression [25],

$$A_2^{ij} = \partial_x \left(g^{ij}\partial_x + c_k^{ij} a_x^k \right) \partial_x. \tag{16}$$

Here and below we omit the index 2 in the notation of metric coefficients $g_{ik}^2(\mathbf{a})$ and connection coefficients $c_{2k}^{ij}(\mathbf{a})$. In [13] (using results from [25]) it was proved that this operator $\hat{A}_2$ is Hamiltonian (ie skew-adjoint and with vanishing Schouten bracket) if and only if the following system is satisfied:

$$c_{skm} = \frac{1}{3}(g_{sm,k} - g_{sk,m}), \tag{17a}$$

$$g_{mk,s} + g_{ks,m} + g_{ms,k} = 0, \tag{17b}$$

$$c_{msk,l} = -g^{pq} c_{pml} c_{qsk}. \tag{17c}$$

where $c_{ijk} = g_{iq}g_{jp}c_k^{pq}$. These conditions are invariant under a class of reciprocal transformations that include affine transformations of the flat coordinates of the last connection. Note that metrics fulfilling (17b) are Monge metrics of quadratic line complexes [3, 13].

Finding a Hamiltonian formulation that involves a third-order operator $\hat{A}_2$ for a hydrodynamic type system is not simple because the Hamiltonian density will be non-local (see (3) and below the Hamiltonian density h_1) in hydrodynamic variables a^k. However, the WDVV hydrodynamic type systems are systems of conservation laws (see (5))

$$a_t^i = (v^i(\mathbf{a}))_x. \tag{18}$$

This means that after a potential substitution $a^i = b_x^i$ we obtain the *nonlinear* system $b_t^i = v^i(\mathbf{b}_x)$ and the Hamiltonian can be obtained by solving the following system of PDEs:

$$v^i(\mathbf{b}_x) = -(g^{ij}(\mathbf{b}_x)\partial_x + c_k^{ij}(\mathbf{b}_x)b_{xx}^k)\frac{\delta \mathbf{H}}{\delta b^j}. \tag{19}$$

In the field variables b^k the Hamiltonian density becomes[2] *local* (see detail in Subsection 3.5).

[2] In general such a density depends polynomially on the independent variable x. However, in the cases considered in this paper the Hamiltonian densities do not depend explicitly on x.

2.2 Reconstructing the Second Hamiltonian Operator

The metric coefficients $g_{ik}(\mathbf{a})$ completely determine a Dubrovin–Novikov type third order Hamiltonian operator $\hat{A}_2$. Indeed, the connection coefficients $c_{ijk}(\mathbf{a})$ are expressible via metric coefficients (see (17a)), while the metric coefficients must fulfill the Potemin system (here we preserved (17b) and substituted (17a) into (17c))

$$g_{mk,s} + g_{ks,m} + g_{ms,k} = 0,$$

$$g_{mk,sl} - g_{ms,kl} = -\frac{1}{3} g^{pq}(g_{pl,m} - g_{pm,l})(g_{qk,s} - g_{qs,k}). \tag{20}$$

Thus if some candidate to be metric coefficients $g_{ik}(\mathbf{u})$ are found in an arbitrary coordinate system u^k, one must look for point transformations $a^k(\mathbf{u})$ such that the metric coefficients $g_{ik}(\mathbf{a}) = g_{ms}(\mathbf{u})\frac{\partial u^m}{\partial a^i}\frac{\partial u^s}{\partial a^k}$ will satisfy the Potemin system. Theoretically this is a very complicated task. However, for particular cases, for instance, for non-diagonalizable hydrodynamic type systems this is an algorithmically solvable problem.

Our main observation is: if an integrable hierarchy of evolutionary equations (12) contains a commuting flow, which is a hydrodynamic type system (see, for instance, (5), (18)), then:

1. its conservation law densities are quasi-homogeneous polynomials (i.e. they are homogeneous polynomials with respect to any derivatives of "x", but coefficients could depend on the field variables u^k in an arbitrary way). For instance, the first two higher conservation law densities of them have the form[3]

$$h_1 = a_{sm}(\mathbf{u})u_x^s u_x^m, \quad h_2 = a_{ms}^{(1)}(\mathbf{u})u_{xx}^m u_{xx}^s + a_{lms}^{(2)}(\mathbf{u})u_{xx}^l u_x^m u_x^s + a_{lsmp}^{(3)}(\mathbf{u})u_x^l u_x^s u_x^m u_x^p. \tag{21}$$

2. the commuting flow (13) becomes[4]

$$u_t^i = (g_2^{ip}\partial_x^3 + \text{l.o.t.})(-2a_{pm}u_{xx}^m + \text{l.o.t.}) = (g_1^{ip}\partial_x + \text{l.o.t.})(2a_{pm}^{(1)}u_{xxxx}^m + \text{l.o.t.}). \tag{22}$$

Then we obtain the relationship in highest order terms (the coefficient of u_{xxxxx}^m)

$$g_2^{ip}a_{pm} = -g_1^{ip}a_{pm}^{(1)}. \tag{23}$$

Thus if the metric coefficients $g_1^{ik}(\mathbf{u})$ are known and $\det a_{jm} \neq 0$, then the metric coefficients of the second operator can be found by

$$g_2^{ij} = -g_1^{ip}a_{pm}^{(1)}c^{mj}, \tag{24}$$

where $a_{im}c^{mj} = \delta_i^j$ and $c^{ip}a_{pj} = \delta_j^i$.

Indeed the commuting hydrodynamic type system has a local Hamiltonian structure (see (14))

$$u_{t^{-1}}^i = A_1^{is}\frac{\delta \mathbf{H}_0}{\delta u^s},$$

where the Hamiltonian density $h_0(\mathbf{u})$ depends on field variables u^k only. Then the next commuting flow

$$u_{t^0}^i = A_2^{is}\frac{\delta \mathbf{H}_0}{\delta u^s} = A_1^{is}\frac{\delta \mathbf{H}_1}{\delta u^s}$$

is an evolutionary system of *third* order (see (15)). In view of the homogeneity of the operators $\hat{A}_1$ and $\hat{A}_2$, this means that h_1 can be chosen in above (left) form (21) up to total x-derivatives:

$$\tilde{h}_1 = \tilde{a}_{sm}(\mathbf{u})u_x^s u_x^m + b_s(\mathbf{u})u_{xx}^s = [\tilde{a}_{sm}(\mathbf{u}) - b_{s,m}(\mathbf{u})]u_x^s u_x^m + (b_s(\mathbf{u})u_x^s)_x.$$

[3] any conservation law density is determined up to a total x-derivative

[4] Here by "l.o.t." we mean "lower order terms".

Then next commuting flow (13) is an evolutionary system of *fifth* order. This means that h_2 can be chosen in above (right) form (21).

Thus we constructed a link between metric coefficients g_1^{ip}, g_2^{ip} and coefficients $a_{sm}(\mathbf{u}), a_{ms}^{(1)}(\mathbf{u})$ in (21) only. This means: if one knows a metric g_1^{ip} and two conservation law densities h_1, h_2, then metric coefficients g_2^{ip} can be found from (24).

In general, any integrable hydrodynamic-type system of (nonlinear) evolutionary PDEs possesses infinitely many local conservation laws of arbitrary order with respect to higher derivatives of the field variables u^k (of the independent variable "x" only)

$$(h(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, ...))_t = (f(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, ...))_x.$$

However in practice conservation laws like in (21) cannot be easily found even in three-component case; in the six component case the direct search of such conservation law densities is probably impossible by any computer algebra system on existing workstations. Nevertheless, this problem is effectively solvable if a Lax pair is known. In such a case the complexity of computation is determined by the complexity of a Taylor expansion (see details below).

In general any integrable system of nonlinear evolutionary PDEs has at least a finite number of linearly independent hydrodynamic conservation law densities $h(\mathbf{u})$. In the non-diagonalizable case (like the WDVV systems), if the system is endowed by a first-order Dubrovin-Novikov homogeneous Hamiltonian operator, the dimension of the space of hydrodynamic integrals is $n + 2$, where n is the number of components [14]. Flat coordinates for the Hamiltonian operator can be selected among the hydrodynamic integrals (see details in [23]). For instance, if $N = 3$, then WDVV associativity equation reduces to a three-component non-diagonalizable hydrodynamic type system which has just *five* hydrodynamic conservation law densities (see details in [10]); if $N = 4$, then the WDVV associativity equations reduce to a compatible pair of six-component non-diagonalizable hydrodynamic type (see (5) and (10)) which have just *nine* hydrodynamic conservation law densities, i.e. flat coordinates of the first Hamiltonian structure $u^0 = a^1, u^1, u^2, u^3, u^4, u^5 = a^4$, a momentum density a^3 quadratic in field variables u^k and two Hamiltonian densities a^5, a^6 rational expressions with respect to these flat coordinates. In these flat coordinates $g_1^{ij} = K^{ij}$ is a constant symmetric non-degenerate matrix. Thus[5] (see (24))

$$g^{ij}(\mathbf{u}) = -K^{ip} a_{pm}^{(1)} c^{mj}. \tag{25}$$

Once the metric coefficients $g^{ij}(\mathbf{u})$ of the contravariant metric are found we want to prove that the covariant metric $g_{ik}(\mathbf{a}) = g_{ms}(\mathbf{u}) \frac{\partial u^m}{\partial a^i} \frac{\partial u^s}{\partial a^k}$ is a Monge metric. To do that one should first find Casimir densities a^k of a Dubrovin–Novikov type third order Hamiltonian operator $\hat{A}_2$. In an arbitrary coordinate system u^k the metric coefficients $g^{ij}(\mathbf{u})$ cannot completely determine such a Hamiltonian operator $\hat{A}_2$ (see (15)), even if all other coefficients are connected with each other via skew-symmetry and Jacobi identity conditions. Thus finding the Casimir densities $a^k(\mathbf{u})$ is really an important problem. Theoretically they can be found easily, because they belong to a finite number of hydrodynamic conservation law densities, which we already discussed above. However, in our case, we have natural candidates to be Casimir densities. Indeed, below we prove that Casimir densities $a^k(\mathbf{u})$ of a Dubrovin–Novikov type third order Hamiltonian operator are precisely coordinates a^k from (5), (6), whose relationship with flat coordinates u^s of a Dubrovin–Novikov type first order Hamiltonian operator is given according to the Viète formulae by (17b).

In our case non-diagonalizable hydrodynamic type systems (5), (10) admit four infinite sequences of homogeneous conservation law densities. The sequences can be deduced from the Lax pair by a standard technique known in integrable systems as follows. The members of such sequences of degree 2 (*i.e.*, whose densities are quadratic in velocities) fulfill a nondegeneracy hypothesis and are enough to reconstruct the leading term of the Hamiltonian operator $\hat{A}_2$.

[5] here again and everywhere below we omit the index 2 of the metric g_2^{ik}.

By eliminating ψ_1, ψ_2, ψ_3 from (6) we obtain the single linear PDE

$$\lambda^2\frac{a^3}{a^1}\psi_{xx}+\lambda^3\left(a^5-\frac{a^2a^3}{a^1}\right)\psi_x+\lambda^4\left(a^6-\frac{(a^3)^2}{a^1}\right)\psi$$
$$=\left(\frac{1}{a^1}\psi_{xx}-\lambda\frac{a^2}{a^1}\psi_x-\lambda^2\frac{a^3}{a^1}\psi\right)_{xx}+\left[\lambda^3\left(\frac{a^2a^3}{a^1}-a^5\right)\psi+\lambda^2\left(\frac{(a^2)^2}{a^1}-a^4\right)\psi_x-\lambda\frac{a^2}{a^1}\psi_{xx}\right]_x.$$

The substitution $\psi=\exp\int rdx$ yields a nonlinear ordinary differential equation on the function r and its first, second and third order derivatives. This function r plays the role of a generating function of conservation law densities with respect to the parameter λ for both systems (5). The expansion of r at infinity (i.e. $\lambda\to\infty$)

$$r=\lambda h_{-1}+h_0+\frac{h_1}{\lambda}+\frac{h_2}{\lambda^2}+...,$$

in the above equation leads to a sequence of differential relationships between the coefficients h_{-1}, h_0, h_1,... The leading term (the coefficient of λ^3) coincides with the characteristic equation of the eigenvalues of the matrix $\mathbf{A}$ (see (7)). Thus the expansion r with respect to the parameter λ has four branches of conservation law densities, where we identify h_{-1} to each of four roots u^k of characteristic polynomial (7), correspondingly. So, changing coordinates a^k to flat coordinates u^m (see (8), (9), (10), (11)) in these expansions we have four branches of conservation law densities, *i.e.*

$$r_{(k)}=\lambda u^k+h_{0k}[\mathbf{u}]+\frac{h_{1k}[\mathbf{u}]}{\lambda}+\frac{h_{2k}[\mathbf{u}]}{\lambda^2}+...,\quad k=1,2,3,4,$$

whose coefficients have more or less compact form (in comparison with expressions $h_{ik}[\mathbf{a}]$ in original coordinates $a^k(\mathbf{u})$) and depend on field variables u^k as well as their higher derivatives with respect to the independent variable "x".

The expressions of such conservation law densities are quasihomogeneous polynomials of degrees $\deg h_{ik}=i+1$ with respect to the grading $\deg u=0$, $\deg\partial_x=1$, and their coefficients are expressible via rational functions of these field variables u^k. We computed (using Reduce [27]) all expressions of h_{ik}, for $k=1$, 2, 3, 4 and $i=0$, 1, 2, 3, the non-trivial ones are of the form (cf. (21))

$$h_{1k}=-\frac{1}{2}G_{ksm}(\mathbf{u})u^s_xu^m_x, \tag{26}$$
$$h_{3k}=Q^{(1)}_{kms}(\mathbf{u})u^m_{xx}u^s_{xx}+Q^{(2)}_{klms}(\mathbf{u})u^l_{xx}u^m_xu^s_x+Q^{(3)}_{klsmp}(\mathbf{u})u^l_xu^s_xu^m_xu^p_x \tag{27}$$

Finding the expressions is not very heavy, but the results need simplification. Indeed, simplification of the rational expressions of the coefficients of derivatives consumes the greatest part of computing resources (18GB of RAM and 6 hours of CPU time). Even after the simplification, the expressions of the coefficients are huge and it is not worth to write them down. This is not a problem as the task is to find the Hamiltonian structure which is checkable by pen and paper (after it has been found!), as one can see below.

We recall that in the case $N=3$ the above procedure leads to 3 conservation law densities h_{1k} ($k=1,2,3$), and that their sum is zero [10]. We have a similar result in our case $N=4$.

Proposition 1 *The conservation law densities $h_{11}(\mathbf{u})$, $h_{12}(\mathbf{u})$, $h_{13}(\mathbf{u})$ are linearly independent, and we have $\sum_{k=1}^4 h_{1k}(\mathbf{u})=0$.*

Proof. The proof is trivial using a computer algebra system. ∎

Remark 2 *The conservation law densities h_{0k} and h_{2k} are trivial for three-component WDVV associativity equations in [10], and h_{0k} is trivial in our six-component hydrodynamic type systems (the proof of triviality is done by CDE [27]). For instance, in the three-component WDVV case we have*

$$h_{01}=-\frac{1}{2}\partial_x\ln(u^1-u^2)(u^1-u^3).$$

However, we have no general proof that this holds for all even-indexed conservation law densities in the above expansion or moreover for all higher WDVV associativity equations.

Now, we show that the above sequence of conservation law densities (26), (27) enables us to reconstruct the metric coefficients $g^{ik}(\mathbf{a})$ of the leading term ∂_x^3 in the second Hamiltonian operator $\hat{A}_2$. Below we formulate the Theorem, whose validity is general and is not limited to our particular hydrodynamic type systems of PDEs. However, this Theorem is based on the interplay between two coordinate systems, i.e. flat coordinates $u^k(\mathbf{a})$ of a first local homogeneous Hamiltonian structure of first order and Casimirs $a^k(\mathbf{u})$ of a second local homogeneous Hamiltonian structure of third order (see again (8), (9), (10), (11)). In our case, for any fixed $i = 1, 2, 3, 4$ the matrices $G_{ijk}(\mathbf{u})$ are degenerate. However, we can introduce the matrix $\tilde{\mathbf{G}}$ whose coefficients are $\xi^m G_{mpq}$, where ξ^k are arbitrary constants. Without loss of generality we may choose any constants ξ^k such that the matrix $\tilde{\mathbf{G}}$ is non-degenerate; then we can introduce the inverse matrix $\mathbf{C}$ whose coefficients we denote C^{km}.

Theorem 3 *Let $\hat{A}_1$ and $\hat{A}_2$ be two Dubrovin–Novikov type Hamiltonian operators of first and of third order, respectively, for hydrodynamic type systems (2), (5), written in flat coordinates u^k, i.e. $\hat{A}_1 = \mathbf{K}\partial_x$.*

Let $\mathbf{H}_{1k} = \int h_{1k}dx$ and $\mathbf{H}_{2k} = \int h_{3k}dx$ be two sets of homogeneous conservation law functionals of degrees 2 and 4, respectively, for any k in a given range;

Then the metric coefficients $g^{ik}(\mathbf{u})$ of a Dubrovin–Novikov type third order Hamiltonian operator $\hat{A}_2$ are uniquely determined by the formula

$$g^{ij}(\mathbf{u}) = 2\xi^m K^{ip} Q^{(1)}_{mpq} C^{qj}. \tag{28}$$

Proof. Let us consider the recurrence relation on $\mathbf{H}_{1k}$ and $\mathbf{H}_{2k}$

$$A_1^{im}\frac{\delta \mathbf{H}_{2k}}{\delta u^m} = A_2^{im}\frac{\delta \mathbf{H}_{1k}}{\delta u^m} \tag{29}$$

(see equations (26), (27)). Then the variational derivatives can be rewritten as

$$\frac{\delta \mathbf{H}_{1k}}{\delta u^j} = G_{kjm}u^m_{xx} + \text{lower order terms},$$

$$\frac{\delta \mathbf{H}_{2k}}{\delta u^j} = 2Q^{(1)}_{kjm}u^m_{xxxx} + \text{lower order terms}.$$

Then we have

$$\begin{aligned} A_1^{ij}\frac{\delta \mathbf{H}_{2k}}{\delta u^j} &= K^{ij}\partial_x(2Q^{(1)}_{kjm}u^m_{xxxx} + \text{lower order terms}) \\ &= 2K^{ij}Q^{(1)}_{kjm}u^m_{xxxxx} + \text{lower order terms}, \\ A_2^{ij}\frac{\delta \mathbf{H}_{1k}}{\delta u^j} &= A_2^{ij}(G_{kjm}u^m_{xx} + \text{lower order terms}) \\ &= g^{ij}G_{kjm}u^m_{xxxxx} + \text{lower order terms}. \end{aligned}$$

By equating the coefficients of highest order derivatives we have $2K^{im}Q^{(1)}_{kms} = g^{im}G_{kms}$ for $k = 1, 2, 3, 4$. By taking the linear combination with coefficients ξ^k of the above identities we can invert the matrix $\xi^k G_{kpq}$ on the right-hand side, which yields the result (28) (cf. (25)). The Theorem is proved. ■

The sequence of homogeneous conservation law densities for our hydrodynamic type systems (5) fulfills the nondegeneracy condition which is necessary in the above Theorem.

Proposition 4 *The following linear combination of the four conservation law densities of degree 2:*

$$C_{ij} = G_{1ij} + \frac{1}{3}G_{2ij} + \frac{1}{2}G_{4ij}$$

fulfills $\det(C_{ij}) \neq 0$.

Proof. The proof is trivial using a computer algebra system. ■

We observe that the proof would be very lengthy by pen and paper: the core computation would be a determinant of a 6×6 matrix whose entries are rational expressions of degree up to 6 in both the numerator and the denominator.

Remark 5 *One might be tempted to use the recurrence relation (29) by a simpler conservation law density. Indeed, the first nontrivial densities are the hydrodynamic type densities. However, this choice on the right-hand side of (29) leads to a trivial identity or to an identity which is not related with the leading coefficient g^{ij}. So, the first 'useful' conservation law densities to our purposes are exactly those quadratic in first derivatives.*

Corollary 6 *If the hydrodynamic type systems (5) admit a second Dubrovin–Novikov type third-order Hamiltonian operator $\hat{A}_2$, then the metric coefficients $g^{ik}(\mathbf{u})$ (i.e. the coefficients of the leading term $g^{ik}\partial_x^3$) have the form which is specified by formula (28) with G_{kjm} and $Q^{(1)}_{kjm}$ found in (26) and (27), and here we chosen $\xi^1 = 1$, $\xi^2 = 1/3$, $\xi^3 = 0$, $\xi^4 = 1/2$.*

The expression of g^{ij} can be found using results from Theorem 3. The explicit computation can be carried out by Reduce [27]. Here most computational resources are consumed by the simplification of the final rational expression of g^{ij}. Even after simplification the expression in coordinates u^k (flat coordinates of the Dubrovin–Novikov type first order Hamiltonian structure) is huge and it is not worth writing it here.

3 Integrability of the Potemin System

In this Section we consider the integrability of Potemin system (17a), (17b), (17c) following from skew-symmetry and Jacobi identity of homogeneous third order Hamiltonian operators (16).

Let introduce the linear expressions

$$\psi^\gamma_k = \psi^\gamma_{km} a^m + \omega^\gamma_k, \tag{30}$$

where ψ^γ_{km} and ω^γ_k are such constants that the matrix $\boldsymbol{\psi}$ is nondegenerate and

$$\psi^\gamma_{km} = -\psi^\gamma_{mk}, \tag{31}$$

$$\sum_{\gamma=1}^n (\psi^\gamma_{is}\psi^\gamma_{jk} + \psi^\gamma_{js}\psi^\gamma_{ki} + \psi^\gamma_{ks}\psi^\gamma_{ij}) = 0, \quad \sum_{\gamma=1}^n (\omega^\gamma_i\psi^\gamma_{jk} + \omega^\gamma_j\psi^\gamma_{ki} + \omega^\gamma_k\psi^\gamma_{ij}) = 0. \tag{32}$$

Theorem 7 *[GVPotemin 2001] The metric coefficients $g_{ik}(\mathbf{a})$ of Potemin system (17a), (17b), (17c) can be presented in the form*

$$g_{mk} = \sum_{\gamma=1}^n \psi^\gamma_m \psi^\gamma_k. \tag{33}$$

Proof. Taking into account (31) and (33), the first equation (17a) of the Potemin system leads to

$$c_{ijk} = -\sum_{\gamma=1}^n \psi^\gamma_i \psi^\gamma_{jk} \tag{34}$$

while the second equation (17b) of the Potemin system implies a set of constraints

$$\sum_{\gamma=1}^n (\psi^\gamma_i\psi^\gamma_{jk} + \psi^\gamma_j\psi^\gamma_{ki} + \psi^\gamma_k\psi^\gamma_{ij}) = 0, \tag{35}$$

which under the substitution $\psi_k^\gamma = \psi_{km}^\gamma a^m + \omega_k^\gamma$ yields (32). Then introducing the inverse metric

$$g^{mk} = \sum_{\gamma=1}^{n} \psi_\gamma^m \psi_\gamma^k \tag{36}$$

such that $\psi_\gamma^i \psi_k^\gamma = \delta_k^i$ and $\psi_\gamma^m \psi_m^\beta = \delta_\gamma^\beta$, substitution of (33) and (34) into the third equation (17c) of the Potemin system yields the identity. ■

Corollary 8 *In the general (n component) case constraints (32) can be resolved. For instance, if $n = 3$, then these constraints reduce to a single equation, i.e. to a sole equation in r.h.s. of (32); if $n = 4$, then the constraints reduce to a system of five equations, which contains a sole equation from l.h.s. of (32) and four other equations from r.h.s. of (32).*

Corollary 9

$$\det g_{ik} = (\det \psi_m^\gamma)^2.$$

3.1 Decomposition of the Monge metric

In this Subsection we present an effective approach for the metric decomposition (33). The linear system of PDEs

$$\psi_{j,k} = \frac{1}{3}\psi_p g^{pq}(g_{qj,k} - g_{qk,j}) \tag{37}$$

on n functions $\psi_k(\mathbf{a})$ can be interpreted as n commuting linear systems of ODEs for each fixed index k. Thus this linear system possesses a general solution parametrized by n arbitrary constants only.

It is easy to see that $(\psi_{j,k})_m = (\psi_{j,m})_k = 0$. This means that ψ_k are *linear* functions with respect to field variables a^s.

Under a geometric viewpoint, the system (37) is equivalent to $\nabla\psi = 0$, where ∇ is the linear connection determined by c_{ijk}. Since the nonlinear condition of the Potemin system is just the requirement of flatness of ∇, the system (37) is just the equation of parallel vectors of ∇. Such an equation always admits n independent solutions which, in our case, are linear, thus making their search particularly simple.

Let us choose n particular solutions ψ_k^γ of this system

$$\psi_{j,k}^\gamma = \frac{1}{3}\psi_p^\gamma g^{pq}(g_{qj,k} - g_{qk,j})$$

such that $\det \psi_p^\gamma \neq 0$. Then taking into account the constraints (35), the metric coefficients can be decomposed in the form (33), (36). The antisymmetric condition (31) can be easily obtained from the above linear system by permutation of the indices j, k. Since ψ_k^γ are linear functions of a^k, then one can identify $\psi_{j,k}^\gamma = \psi_{jk}^\gamma$ which are skewsymmetric constants with respect to lower indices j, k.

Remark 10 *For better convenience the metric decomposition formulas (33), (36) can be slightly modified in the following way. For any Monge metric satisfying the Potemin system* (17a), (17b), (17c) *one can choose any new linear combination of elementary solutions ψ_p^γ, because (37) is a linear system. Thus once some n particular solutions determine a non-degenerate matrix $\boldsymbol{\psi}$, one can introduce the constant non-degenerate matrix $\boldsymbol{\phi}$ such that (cf. (33))*

$$g_{ij} = \phi_{\beta\gamma}\psi_i^\beta \psi_j^\gamma, \tag{38}$$

where $\phi_{\beta\gamma}$ are elements of the matrix $\boldsymbol{\phi}$ and ψ_p^γ are elements of the matrix $\boldsymbol{\psi}$. Indeed, for any symmetric constant matrix $\boldsymbol{\phi}$ we have $\boldsymbol{\phi} = \mathbf{J}\boldsymbol{\Lambda}\mathbf{J}^T$, where $\boldsymbol{\Lambda}$ is a diagonal matrix and $\mathbf{J}$ is an appropriate constant matrix. Introducing the new set of particular solutions $\tilde{\boldsymbol{\psi}} = \mathbf{J}\boldsymbol{\psi}$, the above formula $g_{ij} = \phi_{\beta\gamma}\psi_i^\beta\psi_j^\gamma$ reduces *to the form* $g_{ij} = \lambda_\beta \delta_{\beta\gamma} \tilde{\psi}_i^\beta \tilde{\psi}_j^\gamma$*, where* λ_β *are diagonal elements of the matrix* $\mathbf{\Lambda}$*. Finally, by appropriately scaling* $\tilde{\psi}_i^\beta$*, one can obtain again the original formula (33).*

In this extended construction the connection coefficients (34) become $c_{ijk} = -\phi_{\beta\gamma}\psi_i^\beta \psi_{jk}^\gamma$*, the skew-symmetry condition (31) is the same, but (32) takes the form*

$$\phi_{\beta\gamma}(\psi_{is}^\beta \psi_{jk}^\gamma + \psi_{js}^\beta \psi_{ki}^\gamma + \psi_{ks}^\beta \psi_{ij}^\gamma) = 0, \quad \phi_{\beta\gamma}(\omega_i^\beta \psi_{jk}^\gamma + \omega_j^\beta \psi_{ki}^\gamma + \omega_k^\beta \psi_{ij}^\gamma) = 0. \tag{39}$$

Since the matrices $\boldsymbol{\phi}$ *and* $\boldsymbol{\psi}$ *are non-degenerate, the inverse metric (cf. (36))*

$$g^{ij} = \phi^{\beta\gamma}\psi_\beta^i \psi_\gamma^j \tag{40}$$

can be easily reconstructed.

3.2 Factorised Third Order Homogeneous Hamiltonian Structure

Once a metric decomposition is found, the corresponding Dubrovin–Novikov type differential-geometric third order Poisson bracket (see (16))

$$\{a^i(x), a^j(x')\}_2 = \partial_x(g^{ij}\partial_x + c_k^{ij} u_x^k)\delta'(x - x')$$

also can be written in the factorised form

$$\{a^i(x), a^j(x')\}_2 = \phi^{\beta\gamma}\partial_x \psi_\beta^i \partial_x \psi_\gamma^j \delta'(x - x').$$

This means that any evolutionary system equipped by a Dubrovin–Novikov type third order Hamiltonian structure can be written in the form

$$a_t^i = \partial_x(g^{is}\partial_x + c_k^{is} a_x^k)\partial_x \frac{\delta \mathbf{H}}{\delta a^s} = \phi^{\beta\gamma}\partial_x \psi_\beta^i \partial_x \psi_\gamma^s \partial_x \frac{\delta \mathbf{H}}{\delta a^s}.$$

Thus a Dubrovin–Novikov type third order Hamiltonian operator (see (38))

$$A_2^{ij} = \phi^{\beta\gamma}\partial_x \psi_\beta^i \partial_x \psi_\gamma^j \partial_x \tag{41}$$

can be found together with metric decomposition formulae (38), (40) in our six-component case (see next Section).

Below we provide explicit expressions for *nonlocal* Casimirs and the momentum density; then we reconstruct Hamiltonians for non-diagonalizable hydrodynamic type systems equipped by a Dubrovin–Novikov type third order Hamiltonian operator (41).

3.3 Nonlocal Casimirs

Under the potential substitution $a^i = b_x^i$ the evolutionary conservative system

$$a_t^i = (V^i(\mathbf{a}, \mathbf{a}_x, \mathbf{a}_{xx}, ...))_x \tag{42}$$

takes the form

$$b_t^i = V^i(\mathbf{b}_x, \mathbf{b}_{xx}, ...). \tag{43}$$

Correspondingly, if this evolutionary system has a Dubrovin–Novikov type third order Hamiltonian structure

$$a_t^i = \partial_x(g^{is}\partial_x + c_k^{is} a_x^k)\partial_x \frac{\delta \mathbf{H}}{\delta a^s} = \phi^{\beta\gamma}\partial_x \psi_\beta^i \partial_x \psi_\gamma^s \partial_x \frac{\delta \mathbf{H}}{\delta a^s},$$

then in potential variables b^i we see that evolutionary system (43) has a local Hamiltonian structure of *first* order

$$b^i_t = -(g^{is}\partial_x + c^{is}_k b^k_{xx})\frac{\delta \mathbf{H}}{\delta b^s} = -\phi^{\beta\gamma}\psi^i_\beta \partial_x \psi^s_\gamma \frac{\delta \mathbf{H}}{\delta b^s}, \tag{44}$$

but this is **not** a Dubrovin–Novikov type *first* order Hamiltonian structure, because its coefficients $g^{is}(\mathbf{b}_x)$, $c^{is}_k(\mathbf{b}_x)$, $\psi^i_\beta(\mathbf{b}_x)$ have no geometrical interpretation (i.e. they cannot change under arbitrary point transformations $\tilde{b}^i(\mathbf{b})$ as components of some tensors).

Field variables a^k are Casimirs densities. However, the crucial difference between a Dubrovin–Novikov type first order Hamiltonian structure and Dubrovin–Novikov type third order Hamiltonian structure is existence of n extra Casimirs. Such an observation was first made in [10] where they were found for the three-component non-diagonalizable hydrodynamic type system $a^1_t = a^2_x$, $a^2_t = a^3_x$, $a^3_t = [(a^2)^2 - a^1 a^3]_x$. Under the potential substitution $a^k = b^k_x$ this hydrodynamic type system becomes $b^1_t = b^2_x$, $b^2_t = b^3_x$, $b^3_t = (b^2_x)^2 - b^1_x b^3_x$. However, one can choose three new Casimirs $\mathbf{S}^\beta = \int s^\beta dx$ such that $s^1 = b^1, s^2 = b^2, s^3 = b^3 + b^2 b^1_x$. Then the above nonlinear system reduces again to the conservative form $s^1_t = s^2_x$, $s^2_t = (s^3 - s^2 s^1_x)_x$, $s^3_t = (s^2 s^2_x)_x$. Obviously the inverse transformation is $b^1 = s^1$, $b^2 = s^2$, $b^3 = s^3 - s^2 s^1_x$.

Casimirs $\mathbf{S}^\beta = \int s^\beta dx$ generate "zeroth" flows, i.e.

$$0 = (g^{is}\partial_x + c^{is}_k b^k_{xx})\frac{\delta \mathbf{S}^\alpha}{\delta b^s} = \phi^{\beta\gamma}\psi^i_\beta \partial_x \psi^s_\gamma \frac{\delta \mathbf{S}^\alpha}{\delta b^s}. \tag{45}$$

Theorem 11 *The following functionals determine n nonlocal Casimirs:*

$$\mathbf{S}^\alpha = \int \left(\frac{1}{2}\psi^\alpha_{mk} b^k_x + \omega^\alpha_m\right) b^m dx. \tag{46}$$

Proof. Taking into account $\psi^\alpha_{sm} = -\psi^\alpha_{ms}$, variational derivatives are (see (30))

$$\frac{\delta \mathbf{S}^\alpha}{\delta b^s} = \psi^\alpha_s.$$

Then "zeroth" flows (45) imply

$$0 = \phi^{\beta\gamma}\psi^i_\beta \partial_x \psi^s_\gamma \frac{\delta \mathbf{S}^\alpha}{\delta b^s} = \phi^{\beta\gamma}\psi^i_\beta \partial_x \psi^s_\gamma \psi^\alpha_s.$$

Taking into account $\psi^s_\gamma \psi^\alpha_s = \delta^\alpha_\gamma$, one can see that $\phi^{\beta\gamma}\psi^i_\beta \partial_x \delta^\alpha_\gamma = 0$. The Theorem is proved. ∎

Any Hamiltonian system (44) possesses n conservation laws associated with Casimirs:

$$s^\alpha_t = \frac{\partial s^\alpha}{\partial b^k} b^k_t + \frac{\partial s^\alpha}{\partial b^k_x} b^k_{xt} = -\left[\frac{1}{2}\psi^\alpha_{mk} b^m \phi^{\beta\gamma}\psi^k_\beta \left(\psi^s_\gamma \frac{\delta \mathbf{H}}{\delta b^s}\right)_x + \phi^{\alpha\gamma}\psi^s_\gamma \frac{\delta \mathbf{H}}{\delta b^s}\right]_x.$$

3.4 The Momentum

Any Hamiltonian system (44) has the momentum $\mathbf{P} = \int P dx$; this means that[6]

$$b^i_x = -\phi^{\beta\gamma}\psi^i_\beta \partial_x \psi^s_\gamma \frac{\delta \mathbf{P}}{\delta b^s}.$$

Then the momentum density P can be reconstructed, because all variational derivatives are known:

$$\frac{\delta \mathbf{P}}{\delta b^k} = -\phi_{\beta\gamma}\psi^\beta_k \partial_x^{-1} \psi^\gamma_m b^m_x. \tag{47}$$

[6] A momentum defines a translation. This means that we replace the time variable "t" by the space variable "x" and simultaneously we replace the Hamiltonian density h by the momentum density P.

Theorem 12 *The momentum of the Hamiltonian system (44) is*

$$\mathbf{P} = -\int \left(\frac{1}{3}\phi_{\beta\gamma}\omega_q^\beta \psi_{pm}^\gamma b_x^m + \frac{1}{2}\phi_{\beta\gamma}\omega_p^\beta \omega_q^\gamma \right) b^p b^q dx. \tag{48}$$

Proof. Taking into account (30) and $\psi_{sm}^\alpha = -\psi_{ms}^\alpha$ variational derivatives (47) reduce to the form

$$\frac{\delta \mathbf{P}}{\delta b^k} = -\phi_{\beta\gamma}(\psi_{ks}^\beta b_x^s + \omega_k^\beta)\partial_x^{-1}(\psi_{mp}^\gamma b_x^m b_x^p + \omega_m^\gamma b_x^m) = -\phi_{\beta\gamma}\psi_{ks}^\beta \omega_m^\gamma b^m b_x^s - \phi_{\beta\gamma}\omega_k^\beta \omega_m^\gamma b^m.$$

Indeed, variational derivatives of (48) coincide with above expressions, where we utilized the right equation from (39). The Theorem is proved. ∎

Any Hamiltonian system (44) possesses the conservation law of momentum

$$P_t = \frac{\partial P}{\partial b^k} b_t^k + \frac{\partial P}{\partial b_x^k} b_{xt}^k = \left[b^m \omega_m^\beta \psi_\beta^s \frac{\delta \mathbf{H}}{\delta b^s} - \frac{1}{3}\phi_{\beta\gamma}\omega_q^\beta b^q \psi_{km}^\gamma b^m \phi^{\alpha\delta}\psi_\alpha^k \left(\psi_\delta^s \frac{\delta \mathbf{H}}{\delta b^s} \right)_x - Q \right]_x,$$

where Q is a local expression of field variables b^k and all their higher derivatives due to well-known formula: $Q_x = \frac{\delta \mathbf{H}}{\delta b^s} b_x^s$.

3.5 The Hamiltonian

In this Section we restrict our considerations on Hamiltonian non-diagonalizable hydrodynamic type systems only (see (2) and (5)). In Casimir densities a^k they are written in the conservative form (18). Under the potential substitution $a^i = b_x^i$ the hydrodynamic type system (18) becomes (19) (cf. (42) and (43)), which leads to (see (44))

$$b_t^i = v^i(\mathbf{b}_x) = -\phi^{\beta\gamma}\psi_\beta^i \partial_x \psi_\gamma^s \frac{\delta \mathbf{H}}{\delta b^s}.$$

Thus (cf. (47))

$$\frac{\delta \mathbf{H}}{\delta b^k} = -\phi_{\beta\gamma}\psi_k^\beta \partial_x^{-1}\psi_m^\gamma v^m(\mathbf{b}_x). \tag{49}$$

In general, the expressions $\psi_m^\gamma v^m(\mathbf{b}_x) = (\psi_{mk}^\gamma b_x^k + \omega_m^\gamma) v^m(\mathbf{b}_x)$ depend nonlinearly on b_x^k only. However, in this paper we restrict our consideration to the case where these expressions depend on b_x^k *linearly*, i.e. $\psi_m^\gamma v^m(\mathbf{b}_x) = \eta_m^\gamma b_x^m$, where η_m^γ are constant matrices.

Theorem 13 *The non-diagonalizable Hamiltonian hydrodynamic type systems* (5) *have the Hamiltonians*

$$\mathbf{H} = \frac{1}{2}\int (\zeta_{pqm} b_x^m - \phi_{\beta\gamma}\omega_p^\beta \eta_q^\gamma) b^p b^q dx, \tag{50}$$

where ζ_{pqm} are symmetric constant matrices with respect to first two indices p, q such that

$$\zeta_{kpq} = \frac{1}{3}\phi_{\beta\gamma}(\psi_{kp}^\beta \eta_q^\gamma + 2\psi_{qk}^\beta \eta_p^\gamma), \tag{51}$$

where the constant matrix η_q^γ must satisfy the set of constraints

$$\phi_{\beta\gamma}(\psi_{qp}^\beta \eta_k^\gamma + \psi_{kq}^\beta \eta_p^\gamma + \psi_{pk}^\beta \eta_q^\gamma) = 0, \quad \phi_{\beta\gamma}(\omega_p^\beta \eta_q^\gamma - \omega_q^\beta \eta_p^\gamma) = 0. \tag{52}$$

Proof. Taking into account $\psi^\gamma_m v^m(\mathbf{b}_x) = \eta^\gamma_m b^m_x$ and (30), the variational derivatives (49) reduce to the form

$$\frac{\delta \mathbf{H}}{\delta b^k} = -\phi_{\beta\gamma}\psi^\beta_{kq}\eta^\gamma_p b^p b^q_x - \phi_{\beta\gamma}\omega^\beta_k\eta^\gamma_m b^m.$$

On the other hand, the variational derivatives of (50) are

$$\frac{\delta \mathbf{H}}{\delta b^k} = (\zeta_{kpq} - \zeta_{pqk})b^p b^q_x - \frac{1}{2}\phi_{\beta\gamma}(\omega^\beta_k\eta^\gamma_m + \omega^\beta_m\eta^\gamma_k)b^m.$$

Comparing both the above r.h.s. expressions we obtain the system of linear algebraic equations

$$\zeta_{kpq} - \zeta_{pqk} = -\phi_{\beta\gamma}\psi^\beta_{kq}\eta^\gamma_p,$$

Taking into account set of constraints[7] (52), one can verify that solution (51) of the above system is symmetric with respect to the first indices p, q. The Theorem is proved. ■

Remark 14 *The solution (51) is determined up to a total derivative with respect to "x". One can remove auxiliary elements introducing constants ζ_{kpq} in an alternative more effective way*

$$\zeta_{kkm} = 0, \quad \zeta_{kmk} = \zeta_{pkk} = \phi_{\beta\gamma}\psi^\beta_{km}\eta^\gamma_k, \quad \zeta_{pqk} = \frac{1}{3}\phi_{\beta\gamma}(\psi^\beta_{kq}\eta^\gamma_p - \psi^\beta_{pk}\eta^\gamma_q).$$

i.e. *the corresponding Hamiltonian* **H** *in such a case contains less number of terms.*

Any Hamiltonian system (44) possesses the conservation law of energy. For instance, in the case of hydrodynamic type system (18), one can obtain (see (19) and (50) where $\mathbf{H} = \int h(\mathbf{b}, \mathbf{b}_x)dx$)

$$h_t = \frac{\partial h}{\partial b^k}b^k_t + \frac{\partial h}{\partial b^k_x}b^k_{xt} = -\left(\frac{\partial h}{\partial b^k_x}\phi^{\beta\gamma}\psi^k_\beta\partial_x\psi^s_\gamma\frac{\delta \mathbf{H}}{\delta b^s} + \frac{1}{2}g^{ks}(\mathbf{a})\frac{\delta \mathbf{H}}{\delta b^k}\frac{\delta \mathbf{H}}{\delta b^s}\right)_x.$$

4 Second Hamiltonian Structure for the Six-Component WDVV Hydrodynamic Type System

Reconstructing a third-order Hamiltonian operator $\hat{A}_2$ from its leading term is a very complicated task. Indeed, we know that $\hat{A}_2$ is completely determined by its metric coefficients $g^{ik}(\mathbf{a})$ in the canonical form (16) with respect to its Casimirs. But, if Casimirs are unknown, then the operator will contain many 'spurious' terms which come from the choice of the coordinate system, like d^{ij}_k, d^{ij}_{km}, ... and whose determination is also a nontrivial task.

However, in our case the 'initial' coordinates a^k are all hydrodynamic conservation law densities, i.e. a^k depend just on flat coordinates but not on their higher derivatives. Then we can conjecture that a^k are Casimirs densities for the operator $\hat{A}_2$, as it happened in known examples in $N = 3$. This is indeed the case. So, let us interpret the matrix $g_{is}(\mathbf{u})$ from Corollary 6 as the matrix of a covariant pseudo-Riemannian metric.

Theorem 15 *The metric $g_{is}(\mathbf{u})$ determined by (28) is transformed from the coordinates u^k to the coordinates $a^k(\mathbf{u})$ as*

$$g_{ik}(\mathbf{a}) = \begin{pmatrix} (a^4)^2 & -2a^5 & 2a^4 & -(a^1a^4 + a^3) & a^2 & 1 \\ -2a^5 & -2a^3 & a^2 & 0 & a^1 & 0 \\ 2a^4 & a^2 & 2 & -a^1 & 0 & 0 \\ -(a^1a^4 + a^3) & 0 & -a^1 & (a^1)^2 & 0 & 0 \\ a^2 & a^1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

[7] cf. (52) and the second set of constraints in (39).

The metric $g_{ik}(\mathbf{a})$ is a Monge metric satisfying Potemin system (17b), *(20) and it generates a Dubrovin–Novikov type third-order Hamiltonian operator $\hat{A}_2$ in canonical form* (16).

The Monge metric $g_{ij}(\mathbf{a})$ admits the decomposition (38), where[8]

$$\psi_i^\gamma = \begin{pmatrix} 1 & a^5 & a^4 & 0 & 0 & 0 \\ 0 & a^3 & 0 & 1 & a^5 & 0 \\ 0 & -a^2 & 0 & 0 & -a^4 & 1 \\ 0 & 0 & -a^1 & 0 & a^3 & 0 \\ 0 & -a^1 & 0 & 0 & -a^2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad \phi_{\beta\gamma} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}. \tag{53}$$

So the Hamiltonian operator $\hat{A}_2$ can be rewritten in the simplified form (41), where inverse matrices are

$$\psi_\gamma^i = \frac{1}{a^1}\begin{pmatrix} a^1 & 0 & 0 & a^4 & a^5 & a^3a^4 - a^2a^5 \\ 0 & 0 & 0 & 0 & -1 & a^2 \\ 0 & 0 & 0 & -1 & 0 & -a^3 \\ 0 & a^1 & 0 & 0 & a^3 & a^1a^5 - a^2a^3 \\ 0 & 0 & 0 & 0 & 0 & -a^1 \\ 0 & 0 & a^1 & 0 & -a^2 & (a^2)^2 - a^1a^4 \end{pmatrix}, \quad \phi^{\beta\gamma} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

In this case $\det \boldsymbol{\phi} = 1$, $\det \mathbf{g} = (a^1)^4$, $\det \boldsymbol{\psi} = -(a^1)^2$.

Each of the hydrodynamic type systems (5) *admits a Hamiltonian formulation by means of $\hat{A}_2$, with nonlocal Hamiltonian density $\tilde{h}_k(\mathbf{b}, \mathbf{b}_x)$, respectively (see (50))*

$$\tilde{h}_1 = -b^4b^5b^1_x - b^5b^2b^2_x + b^2b^4b^3_x - b^2b^6, \tag{54}$$

$$\tilde{h}_2 = -b^3b^5b^2_x + b^4b^3b^3_x + b^1b^5b^5_x - b^3b^6, \tag{55}$$

where $a^k = b^k_x$. Both hydrodynamic type systems (5) *also have a common momentum $\mathbf{P} = \int P dx$ and the same set of nonlocal Casimirs $\mathbf{S}^k = \int s^k dx$, where (see (48))*

$$P = -b^3b^2b^2_x - b^1b^3b^4_x + b^1b^2b^5_x - b^1b^6 - (b^3)^2,$$

$$s^1 = b^1, \quad s^2 = b^2, \quad s^3 = b^3, \quad s^4 = b^4b^1_x, \quad s^5 = b^5b^1_x + b^3b^2_x, \quad s^6 = b^5b^2_x + b^3b^4_x + b^6.$$

The operators $\hat{A}_1$, $\hat{A}_2$ form a commuting pair, hence the systems (5) *have a bi-Hamiltonian formulation.*

Proof. It is not difficult (using Reduce) to change the coordinates of the metric $g_{ij}(\mathbf{u})$ using the Viète formulae (8). Then, again by Reduce, it is easy to verify that the metric $g_{ij}(\mathbf{a})$ fulfills the condition (17b) which ensures the fact that $g_{ij}(\mathbf{a})$ is a Monge metric, and fulfills the nonlinear equation (17c). This means that the operator $\hat{A}_2$ defined through (16) is a Hamiltonian operator.

The metric decomposition (38), (40) follows from the integrability of linear system (37), which is easily solved by Reduce once the linearity of solutions ψ^i_γ is taken into account. The nonlocal Casimirs $\mathbf{S}^k = \int s^k dx$ and the momentum $\mathbf{P} = \int P dx$ can be found by the formulae (46) and (48), respectively.

The computation of Hamiltonians $\tilde{\mathbf{H}}_k = \int \tilde{h}_k dx$ is slightly more complicated. First of all, one should find constant matrices $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2$ determined by the equations $\psi^\gamma_m v^m(\mathbf{b}_x) = \eta^\gamma_{1m} b^m_x$, $\psi^\gamma_m w^m(\mathbf{b}_x) = \eta^\gamma_{2m} b^m_x$ (see (5)) for both commuting systems $b^i_y = v^i(\mathbf{b}_x)$ and $b^i_z = w^i(\mathbf{b}_x)$. Here they are

$$\boldsymbol{\eta}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\eta}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

[8]In the matrix $P = (\psi^\beta_i)$ the upper index β is a column index, so that if $\boldsymbol{\Phi} = (\phi_{\alpha\beta})$ and $\mathbf{g} = (g_{ij})$ we can rewrite (38) as $\mathbf{g} = P\Phi P^T$ and (44) as $\mathbf{b}_t = -(P^T)^{-1}\boldsymbol{\Phi}^{-1}\partial_x P^{-1}\delta\mathbf{H}/\delta\mathbf{u}$.

The Hamiltonian densities can then be found by the formulae (50).

Finally, it is easy by CDE (but probably impossible by pen and paper) to prove that $[\hat{A}_1, \hat{A}_2] = 0$, ie the first and third order operators commute with respect to the Schouten bracket. This endows both the WDVV systems (5) by the structure of a bi-Hamiltonian system. ■

Remark 16 *Using CDE [27] it is not difficult to prove that $\hat{A}_2(\mathbf{p})$ fulfills the criterion of [18] for the left system (5). This means that $\hat{A}_2$ maps conservation laws to symmetries for that system, and analogously for the right system.*

Remark 17 *The Monge metric $g_{ij}(\mathbf{a})$ is not flat: its Riemann curvature tensor does not vanish. Moreover, its scalar curvature (Ricci scalar) is zero*[9] *but its sectional curvature is non constant. The Monge metric is also not projectively flat: its Weyl tensor is nonzero. The known examples in $N = 3$ have flat metrics: this is a first example of a Dubrovin–Novikov type third-order Hamiltonian operator with a non-trivial curvature.*

5 Conclusion and Outlook

It is worth to make a short discussion in order to establish to what extent our results can be checked by a reader.

While the Monge property of g_{ij} is easily checkable by pen and paper, the hand-verification of the nonlinear condition (17c) looks like it would take much longer.

In principle it is possible to find nonlocal Casimirs, the momentum and the Hamiltonian by pen and paper, using our formulae; of course it is faster to use computer. However, we stress that the decomposition of the Monge metric and the fact that $\tilde{h}_1$ and $\tilde{h}_2$ are Hamiltonians for the respective systems (5) can be easily checked by pen and paper, thus validating our long computer calculations.

The only computation that looks almost impossible to complete by pen and paper is $[\hat{A}_1, \hat{A}_2] = 0$. In this respect CDE proved to yield the correct result for the Schouten bracket of a fairly high number of test cases (see [13, 27]), even in the multidimensional situation[10].

Now, let us discuss what are the perspectives of our research work.

Non-diagonalizable hydrodynamic type systems (2) which possess conservation law densities quadratic in first derivatives (see (21)) were investigated in [14]. Following the authors of that paper we can say that in general a non-diagonalizable hydrodynamic type system (2) is integrable if it possesses ‘sufficiently’ many conservation law densities which are quadratic in first derivatives. It was proved that in the three component case ‘sufficiently’ many means 2; here, in the four component case, this number is 3. In the n-component case this question is open due to its high computational complexity. This problem still is not yet solved. However, we believe that our Conjecture is valid.

Conjecture: *If a non-diagonalizable hydrodynamic type system* (2) *possesses at least one conservation law density which is quadratic in first derivatives and also has a Dubrovin–Novikov type first order Hamiltonian structure, then such a system is integrable by the inverse scattering transform method (this means automatically that such a system has ‘sufficiently’ enough conservation law densities which are quadratic in first derivatives). Moreover then this system has a Dubrovin–Novikov type third order Hamiltonian structure.*

We believe that a consistent subclass of multi-dimensional WDVV associativity equations written as a family of commuting hydrodynamic type systems are bi-Hamiltonian in the above sense. A striking feature of all bi-Hamiltonian examples of WDVV systems is that their third order Hamiltonian operators are uniquely determined by a quadratic line complex [3, 13]. It would be interesting to know if this is

[9] We are grateful to O. Mokhov for communicating us this fact.

[10] We thank M. Casati for providing us examples of commuting and non-commuting bivectors in dimension $2 + 1$.

connected by any means to the rich underlying geometry (and in particular projective geometry) which is connected to Gromov–Witten invariants.

In the aforementioned paper [14], the authors considered the Matrix Hopf equation

$$U_t = (U^2)_x,$$

where U is a symmetric matrix of order $N \times N$, with the further reduction $\mathrm{tr}U^k$ =const, $k = 1, 2, ..., N$. This matrix equation represents a non-diagonalizable hydrodynamic type system of order $n \times n$ where $n = N(N-1)/2$. Such a system possesses $N(N+1)/2 - 1$ hydrodynamic conservation laws and precisely $N - 1$ conservation laws which are quadratic in first derivatives. This matrix Hopf equation is equivalent to the remarkable N wave system (see detail in [9]).

Our hypothesis is that the first commuting flows

$$U_{t^k} = (U^k)_x, \quad k = 2, 3, ..., N-1$$

up to an appropriate reciprocal transformations are equivalent to corresponding multi-dimensional WDVV associativity equations. This problem should be investigated elsewhere.

The case considered in this paper is determined by the particular choice $N = 4$. Indeed, our two six-component commuting non-diagonalizable hydrodynamic type systems have nine hydrodynamic conservation laws and three conservation laws which are quadratic in first derivatives. They are connected with the pair of matrix Hopf equations $U_y = (U^2)_x$, $U_z = (U^3)_x$ (with four constraints $\mathrm{tr}U^k$=const, $k = 1, 2, 3, 4$) by an appropriate reciprocal transformation.

Finally, we remark that the triviality of the Hamiltonian cohomology of $\hat{A}_1$ [15] implies that $\hat{A}_2 = L_\tau \hat{A}_1$. The vector field τ is not uniquely determined; the *mastersymmetry* of the WDVV associativity equations should be found as one of the possible choices for τ [4]. The explicit expression for all possible generalized vector field τ of the above type can be of interest, and is computable via the technique of Lagrangian representation [22]. We solved this problem for the 3-component WDVV [28]; expressions for the 6-component WDVV are available upon request.

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