Absolute contact differentiation on submanifolds of Cartan spaces

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Abstract

We introduce the general concept of higher order absolute contact differentiation that is based on the idea of semiholonomic contact elements. We clarify how the moving frame method leads to the coordinate functions of the field of \( r \)-th order contact elements on a submanifold of Klein space and of the \( r \)-th absolute contact differential of a submanifold of Cartan space. We point out that the standard geometric objects of submanifolds are defined on contact elements, so that they are of universal character. In examples, we use heavily the concept of universal horizontal and vertical bundle over contact elements.

Keywords: semiholonomic contact elements, absolute contact differentiation, submanifolds of Cartan spaces, geometric objects of submanifolds, universal horizontal and vertical bundles.

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The present paper was initiated by a conference talk by the second author on the contact element approach to geometric objects of submanifolds of Riemannian manifolds, [24]. Generally speaking, he pointed out that these objects are of universal character. Indeed, they are defined on the bundles of contact \((n, r)\)-elements, so that they can be applied to every \(n\)-submanifold and are independent of its parametrization. This fact was also observed by the first author for submanifolds of Klein spaces in connection with the Cartan method of moving frames, [14], as well as for submanifolds of Cartan spaces, [13]. (We replace the term “Cartan geometry” from [22] by “Cartan space”, see Section 3 for justification.)

In the course of the present research we realized that our approach to submanifolds of Cartan spaces is essentially based on the ideas of semiholonomic contact element and absolute contact differentiation. So, in the present paper we start with basic properties of nonholonomic and semiholonomic contact elements. In Section 2 we introduce the general concept of \(r\)-th order absolute contact differentiation, that leads to semiholonomic contact elements. Section 3 is devoted to two equivalent definitions of Cartan space. The interrelations between both points of view are essential for our research.

In Section 4 we recall, in the case of an arbitrary Klein space \(S = G/H\), how the moving frame approach leads to the coordinate functions of the field of \(r\)-th order contact elements determined by a submanifold \(N \subset S\). Then we present the general concept of \(r\)-th order geometric object for \(n\)-submanifolds of \(S\) that is motivated by the computational procedures related with the Cartan method of moving frames from [17]. We also clarify that the Cartan prolongation procedure leads to the equations of the infinitesimal action of \(H\) on the standard fiber of the bundle of contact \((n, r)\)-elements on \(S\), that can be used for evaluating the geometric objects. In Section 5 we modify these ideas to submanifolds of a Cartan space \(\mathcal{S}\) of type \(S\). This is based on the concept of semiholonomic \((n, r)\)-object. In Section 6, Proposition 8 reads that if \(\mathcal{S}\) is torsion-free, then the values of the second order absolute contact differentiation are holonomic. In particular, this is true for the submanifolds of a Riemannian manifold, that is considered as a Cartan space with respect to the Levi-Civita connection.
In Section 7 we define the universal horizontal and vertical bundles for $n$-submanifolds. As an example, we discuss the universal version of the fundamental vertical-valued quadratic form for submanifolds of affine spaces. In Section 8 we introduce the concept of reduced torsion and clarify that its universal version coincides with the difference tensor of second order semiholonomic contact elements. This yields another proof of Proposition 8. At this occasion we also illustrate the use of the algorithm from Section 5. Further we point out that in the case of a 2-submanifold of a 3-space with projective connection, the reduced torsion gives rise to an invariant discovered already by É. Cartan in [4]. In the last section, we extend the idea of universality to a wide class of geometric objects for submanifolds.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from [16].

1 Semiholonomic contact elements

The bundle of contact $(n, r)$-elements $K^r_n M$ on a manifold $M$ can be defined as the factor space

$$K^r_n M = \text{reg} T^r_n M / G^r_n, \quad n < m = \dim M$$

of the space of regular $(n, r)$-velocities on $M$ with respect to the right action, determined by the jet composition, of the $r$-th differential group $G^r_n$ in dimension $n$, [16]. So every $n$-submanifold $N \subset M$ defines a contact $(n, r)$-element $k^r_x N$ for every $x \in N$. This gives rise to a map $k^r_x : N \to K^r_n M$, that can be viewed as a section of the restriction $(K^r_n M)_N$ of $K^r_n M$ over $N$. We write $k : \text{reg} T^r_n M \to K^r_n M$ for the factor projection and denote by the same symbol the induced map $k : \text{reg} J^r(N, M) \to K^r_n M$. We remark that these classical contact elements are also discussed by P.J. Olver, [20], and are called jets of submanifolds in [1].

We extend the idea of nonholonomic and semiholonomic jets by C. Ehresmann [5, p. 361] to contact elements. To this aim, we recall the definition of jet prolongations of fibered manifolds, [15]. Let $p : Y \to M$ be a fibered manifold and $J^r Y$ denote the bundle of standard (holonomic) $r$-jets of local sections of $Y$. The $r$-th nonholonomic jet prolongation $\tilde{J}^r Y$ is defined by the induction $\tilde{J}_1^1 Y = J^1 Y$, $\tilde{J}^r Y = \tilde{J}^1(\tilde{J}^{r-1} Y \to M)$. The $r$-th semiholonomic jet prolongation $\check{J}^r Y$ is defined by induction as the space of first-order
jets $j^1_x$s of local sections $s$ of $\tilde{J}^{r-1}Y \rightarrow M$ such that $s(x) = j^1_x(\beta_{r-1} \circ s)$, where $\beta_{r-1}: \tilde{J}^{r-1}Y \rightarrow \tilde{J}^{r-2}Y$ is defined in the induction procedure with $\tilde{J}^1Y = J^1Y \rightarrow Y$. We have the inclusions $J^rY \subset \tilde{J}^rY \subset \bar{J}^rY$. The first one is given by the iteration $j^r_xs \mapsto j^1_x(u \mapsto j^1_x(1)s)$, the second one is straightforward. If $p: Y = M \times N \rightarrow M$ is the first product projection, then we set $\tilde{J}^r(M, N) = \tilde{J}^r(M \times N)$ and $\bar{J}^r(M, N) = \bar{J}^r(M \times N)$.

**Definition 1.** The space $\tilde{K}_n^rM$ of nonholonomic $(n, r)$-elements on $M$ is defined by the iteration $\tilde{K}_n^rM = K_1^1(\tilde{K}_{n-1}^rM)$, $\tilde{K}_n^1M = K_1^1M$.

Hence $\tilde{K}_n^rM \rightarrow \tilde{K}_{n-1}^rM$ is a fibered manifold. The injection $K^r_nM \hookrightarrow \tilde{K}_n^rM$ is determined by the rule

$$k^r_xN \mapsto k^1_X(k^r_{n-1}N), \quad X = k^r_{n-1}N,$$

where $k^r_{n-1}$ is interpreted as a submanifold of $K^r_{n-1}M \subset \tilde{K}_{n-1}^rM$.

We are going to clarify up to what extent the manifold of nonholonomic contact elements can be regarded as a quotient manifold analogously to (1). To this aim, we recall the definition of composition of nonholonomic jets [5], [15]. Let $M$, $N$, $Q$ be three manifolds. For $r = 1$ we have the standard composition of 1-jets in $J^1(N, Q)$ with 1-jets in $J^1(M, N)$, yielding 1-jets in $J^1(M, Q)$. If $\beta: \tilde{J}^{r-1}(M, N) \rightarrow N$ is the target jet projection, $X = j^1_x(s(u)) \in \tilde{J}^r_x(M, N)_y$, $u \in M$, and $Z = j^1_y\sigma \in \tilde{J}^r_y(N, Q)_z$, $y = \beta(s(x))$, then we define

$$Z \circ X = j^1_x(\sigma(\beta(s(u))) \circ s(u)) \in \tilde{J}^r_x(M, Q)_z$$

with the composition of nonholonomic $(r - 1)$-jets on the right hand side. We say that $X \in \tilde{J}^r_x(M, N)_y$ is regular, if there exists $Z \in \tilde{J}^r_y(N, Q)_z$ such that $Z \circ X = j^1_x id_M$. There are $r$ underlying 1-jets of $X$ and $X$ is regular iff all of them correspond to injective linear maps $T_xM \rightarrow T_yN$.

We define $\tilde{T}^r_nM = \tilde{J}^r_0(\mathbb{R}^n, M)$. This is extended into a bundle functor $\tilde{T}^r_n$ on $\mathcal{M}f$ in the standard way, [5]. A natural equivalence of functors

$$\mu^r_n: \tilde{T}^r_nM \rightarrow T^r_n(\tilde{T}^{r-1}_nM)$$

is defined as follows. Every $X \in \tilde{T}^r_nM$ is of the form $X = j^1_0\varphi$, where $\varphi: \mathbb{R}^n \rightarrow \tilde{J}^{r-1}(\mathbb{R}^n, M)$ is a section of the source projection $\tilde{J}^{r-1}(\mathbb{R}^n, M) \rightarrow \mathbb{R}^n$. On the other hand, $Z \in T^1_n(\tilde{T}^{r-1}_nM)$ means $Z = j^1_0\psi$ with $\psi: \mathbb{R}^n \rightarrow \tilde{J}^{r-1}_0(\mathbb{R}^n, M)$. 

Write \( t_u : \mathbb{R}^n \rightarrow \mathbb{R}^n \) for the translation \( x \mapsto x + u \). Then \( u \mapsto (\varphi(u) \circ j_0^{-1} t_u) \) is a map \( \mathbb{R}^n \rightarrow \tilde{J}_0^{r-1}(\mathbb{R}^n, M) \) and we set
\[
\mu^r_M(X) = j_0^1(\varphi(u) \circ j_0^{-1} t_u).
\]
To obtain the inverse map, we consider \( u \mapsto \psi(u) \circ j_u^{r-1} t_u^{-1} \). Then we have
\[
(\mu^r_M)^{-1}(Z) = j_0^1(\psi(u) \circ j_u^{r-1} t_u^{-1}).
\]
One verifies easily that \( \mu^r_M \) maps \( \text{reg} \tilde{T}^r_n M \) into \( \text{reg} T^r_n (\text{reg} \tilde{T}^{r-1}_n M) \).
On every fibered manifold \( p : Y \rightarrow M \), a contact element \( X \in K^1_n Y \) is said to be transversal, if the underlying linear \( n \)-space of \( X \) has zero intersection with the vertical tangent space of \( Y \). We write \( \text{tr} K^1_n Y \subset K^2_n Y \) for the subset of all transversal contact \((n,1)\)-elements on \( Y \). This is an open subset of \( K^1_n Y \) and \( \text{tr} K^1_n Y \rightarrow Y \) is a fibered manifold. For \( r \geq 2 \), the bundle \( \text{tr} \tilde{K}^r_n M \subset \tilde{K}^r_n M \) of nonholonomic transversal contact \((n,r)\)-elements on \( M \) is defined by the iteration
\[
\text{tr} \tilde{K}^r_n M = \text{tr} K^1_n (\text{tr} \tilde{K}^{r-1}_n M \rightarrow M).
\]
We recall that \( \tilde{G}^r_n = \text{reg} \tilde{J}_0^1(\mathbb{R}^n, \mathbb{R}^n)_0 \) is a group with respect to the composition of nonholonomic jets.

**Proposition 1.** We have
\[
(2) \quad \text{tr} \tilde{K}^r_n M = \text{reg} \tilde{T}^r_n M / \tilde{G}^r_n.
\]
We write \( k : \text{reg} \tilde{T}^r_n M \rightarrow \text{tr} \tilde{K}^r_n M \) for the factor projection.

**Proof.** Assume by induction \( \text{tr} \tilde{K}^{r-1}_n M = \text{reg} \tilde{T}^{r-1}_n M / \tilde{G}^{r-1}_n \). So we have defined \( k : \text{reg} \tilde{T}^{r-1}_n M \rightarrow \text{tr} \tilde{K}^{r-1}_n M \). Consider \( X \in \text{reg} T^1_n (\text{reg} \tilde{T}^{r-1}_n M), X = j_0^1 \varphi(u), \varphi : \mathbb{R}^n \rightarrow \tilde{T}^{r-1}_n M \). Then \( u \mapsto k(\varphi(u)) \) is the parametrization of an \( n \)-dimensional submanifold of \( \text{tr} \tilde{K}^{r-1}_n M \) that is transversal to \( \text{tr} \tilde{K}^{r-1}_n M \rightarrow M \). Hence we have \( k(j_0^1 k(\varphi(u))) \in \text{tr} \tilde{K}^r_n M \). Consider another \( \psi(v) : \mathbb{R}^n \rightarrow \tilde{T}^{r-1}_n M \) such that \( k(j_0^1 k(\psi(v))) = k(j_0^1 k(\varphi(u))) \). First of all, there is a map \( v = f(u) \) such that
\[
j_0^1 k(\psi(f(u))) = j_0^1 k(\varphi(u)).
\]
By the induction hypothesis, there is a map \( g : \mathbb{R}^n \rightarrow \tilde{G}^{r-1}_n \) such that \( \psi(f(u)) \circ g(u) = \varphi(u) \). We have \( j_0^1 f \in G^1_n \) and \( j_0^1 g \in T^1_n \tilde{G}^{r-1}_n \) and our construction is in accordance with the well known expression \( \tilde{G}^{r}_n = G^1_n \rtimes T^1_n \tilde{G}^{r-1}_n \), [16]. □
Definition 2. The bundle of semiholonomic contact \((n, r)\)-elements \(\tilde{K}^{r}_n M \subset \tilde{K}^{r}_n M\) is the subset of all \(k^1_X Q\) such that \(Q \subset \tilde{K}^{r-1}_n M\), \(X = k^1_{\beta_{r-1} (X)} (\beta_{r-1} (Q))\), where \(\beta_{r-1}: \tilde{K}^{r-1}_n M \to \tilde{K}^{r-2}_n M\) is defined in the induction procedure starting with the bundle projection \(K^{r}_n M \to M\).

We easily deduce \(\tilde{K}^{r}_n M \subset \text{tr} \tilde{K}^{r}_n M\) and \(K^{r}_n M \subset \tilde{K}^{r}_n M\). Analogously to Proposition 1, we obtain

\[ (3) \quad \tilde{K}^{r}_n M = \text{reg} \tilde{T}^{r}_n M/\tilde{G}^{r}_n, \]

where \(\tilde{G}^{r}_n = \text{reg} \tilde{J}^{r}_0 (\mathbb{R}^n, \mathbb{R}^n)\) is a subgroup of \(\tilde{G}^{r}_n\). The canonical projection to lower order semiholonomic contact elements will be denoted by the same symbol \(\pi^{r}_s\), \(s < r\), as in the jet case.

The underlying contact \((n, 1)\)-element \(X_1\) of \(X \in (K^{r}_n \mathbb{R}^m)_{x}\) is identified with a linear \(n\)-space in \(T_x \mathbb{R}^m\). Write \(\mathbb{R}^{n,m-n}\) for the product bundle \(\mathbb{R}^n \times \mathbb{R}^{m-n}\) and \(\pi: \mathbb{R}^m \to \mathbb{R}^m\) for its bundle projection. Denote by \(\tau K^{r}_n \mathbb{R}^m \subset K^{r}_n \mathbb{R}^m\) the open subset of all \(X\) such that \(X_1\) is transversal to \(\pi\). It is well known that \(\tau K^{r}_n \mathbb{R}^m\) is identified with the jet prolongation \(J^{r} \mathbb{R}^{n,m-n}\), [16].

In the nonholonomic case \(X \in \tilde{K}^{r}_n \mathbb{R}^m\), we have \(r\) underlying contact \((n, 1)\)-elements \(X^{(1)}_1, \ldots, X^{(r)}_1\). Write \(\tau \tilde{K}^{r}_n \mathbb{R}^m \subset \tilde{K}^{r}_n \mathbb{R}^m\) for the open subset of all \(X\) such that all \(X^{(1)}_1, \ldots, X^{(r)}_1\) are transversal to \(\pi\).

**Proposition 2.** \(\tau \tilde{K}^{r}_n \mathbb{R}^m\) is identified with the \(r\)-th nonholonomic prolongation \(J^{r} \mathbb{R}^{n,m-n}\).

**Proof.** By Proposition 1, \(X \in \tau \tilde{K}^{r}_n \mathbb{R}^m\) can be expressed as \(X = Z \circ \tilde{G}^{r}_n\) with \(Z \in \text{reg} \tilde{T}^{r}_n \mathbb{R}^m\). Write \(Z^{(1)}_1, \ldots, Z^{(r)}_1\) for the underlying 1-velocities of \(Z\). Then \(j^1_0 \pi \circ Z^{(1)}_1, \ldots, j^1_0 \pi \circ Z^{(r)}_1\) are invertible 1-jets, so that \(\zeta := (j^1_0 \pi) \circ Z \in \tilde{J}^{r}_0 (\mathbb{R}^n, \mathbb{R}^n)\) is invertible. Then \(Z \circ \zeta^{-1}\) satisfies \((j^1_0 \pi) \circ (Z \circ \zeta^{-1}) = j^r_{\pi (x)} \text{id}_{\mathbb{R}^m}\), which implies \(Z \circ \zeta^{-1} \in \tilde{J}^{r} \mathbb{R}^{n,m-n}\). \(\square\)

In the semiholonomic case \(X \in \tilde{K}^{r}_n \mathbb{R}^m\), we have \(X^{(1)}_1 : \cdots = X^{(r)}_1 = : X_1\). We write \(\tau \tilde{K}^{r}_n \mathbb{R}^m \subset \tilde{K}^{r}_n \mathbb{R}^m\) for the open subset of all \(X\) such that \(X_1\) is transversal to \(\pi: \mathbb{R}^m \to \mathbb{R}^m\). In the same way as in Proposition 2, we construct an identification

\[ (4) \quad \tau \tilde{K}^{r}_n \mathbb{R}^m \approx \tilde{J}^{r} \mathbb{R}^{n,m-n}, \]

where \(\tilde{J}^{r} \mathbb{R}^{n,m-n}\) denotes the \(r\)-th semiholonomic prolongation of \(\mathbb{R}^{n,m-n}\). In particular, for \(n = 1\) we have \(\tilde{K}^{r}_1 M = K^{r}_1 M\).
2 The absolute contact differentiation

Consider a principal bundle $P(M,G)$ with a principal connection $\Gamma$, a left $G$-space $F$ and the associated bundle $E = P[F]$. For every section $s$ of $E$, its absolute differential can be viewed as a section

$$\nabla_\Gamma s: M \rightarrow \bigcup_{x \in M} J^1_x(M,E_x)s(x),$$

[16]. If $\Gamma(u) = j^1_x\rho$ for a local section $\rho$ of $P$, then $(\nabla_\Gamma s)(x)$ is transformed by $\tilde{u}^{-1}$ into

$$j^1_x(\tilde{\rho}(y)^{-1}(s(y))) \in J^1_x(M,F), \quad y \in M,$$

where $\tilde{u}: F \rightarrow E_x$ denotes the frame map corresponding to $u \in P$ (see also the beginning of Section 4).

Having in mind submanifolds of Cartan spaces, we introduce the concept of absolute contact differential. Replace $M$ by $N$ and assume $n = \dim N < \dim F$. Clearly, all spaces $K^1_n(E_x), x \in N$, form an associated bundle

$$\bigcup_{x \in N} K^1_n(E_x) = P[K^1_nF].$$

Assume further that each $\nabla_\Gamma s(x)$ is a regular 1-jet.

**Definition 3.** The absolute contact differential $k\nabla_\Gamma$ of $s$ is defined by

$$((k\nabla_\Gamma)s)(x) = k((\nabla_\Gamma s)(x)), \quad x \in N.$$

Hence $k\nabla_\Gamma s$ is a section $N \rightarrow P[K^1_nF]$. Since we have a section of another bundle associated to $P$, we can construct

$$\nabla_\Gamma ((k\nabla_\Gamma)s): N \rightarrow \bigcup_{x \in N} J^1_x(N,K^1_n(E_x)).$$

This is formed by regular 1-jets and we define

$$((k\nabla^2_\Gamma)s)(x) = k(\nabla_\Gamma ((k\nabla_\Gamma)s)(x)).$$

One verifies easily that this is an element of $\bar{K}^2_n(E_x)$. Hence $(k\nabla^2_\Gamma)s$ is a section $N \rightarrow P[\bar{K}^2_nF]$. 

Definition 4. The \( r \)-th absolute contact differential of \( s \) is defined by the iteration

\[
((k\nabla_s^r)s)(x) = k(\nabla_s((k\nabla_s^{r-1})s))(x).
\]

By the very definition of semiholonomic contact \((n, r)\)-elements, we deduce that (6) form a section

\[
(k\nabla_s^r)s: N \to P[\bar{K}_r^n F].
\]

Remark 1. If \( \Gamma \) is curvature free, then \( P \) can be locally viewed as the product \( N \times G \) with the canonical flat connection. Then (5) implies that the values of \((k\nabla_s^r)s\) are holonomic contact elements for every section \( s: M \to E \).

3 Cartan spaces

We recall that a Klein space is a manifold \( S \) with a transitive left action \((g, x) \mapsto gx\) of a Lie group \( G \). Fix a point \( c \in S \) and write \( H \) for its stability group. Then \( S \) coincides with the coset space \( S = G/H \), \( c = \{H\} \) and \( G \) can be viewed as a principal \( H \)-bundle over \( S \) with bundle projection \( g \mapsto gc \). Every \( g \in G(S, H) \) is interpreted as a frame \( \hat{g}: S \to S, \hat{g}(a) = ga, a \in S \).

A “curved” version of \( S \) can be defined in two formally different ways. First we present the viewpoint from the book by Sharpe, [22]. Consider a pair \((G, H)\) of a Lie group \( G \) and a closed subgroup \( H \subset G \).

Definition 5a. Cartan geometry of type \((G, H)\) is a principal bundle \( Q(M, H) \) with 1-form \( \omega: TQ \to \mathfrak{g} \) (which is said to be Cartan connection) such that

1. \( \omega(u): T_uQ \to \mathfrak{g} \) is a linear isomorphism for every \( u \in Q \),
2. \( (R_h)^*\omega = \text{Ad}(h^{-1}) \circ \omega \) for every \( h \in H \),
3. \( \omega(X^*(u)) = X \) for every \( X \in \mathfrak{h} \) and every \( u \in Q \), where \( X^* \) is the fundamental vector field on \( Q \) induced by \( X \).

We remark that, in addition to [22], further interesting examples of Cartan spaces can be found in [2] and [23].

In what follows we assume \( G \) acts effectively on the coset space \( G/H \). So \( S = G/H \) is a Klein space. Clearly, \( T_cS = \mathfrak{g}/\mathfrak{h} \).

On the other hand, consider \( P(M, G), F, E = P[F] \) as in Section 2 and fix a section \( s: M \to E \). The following definition in [10] or [11] was based directly on some ideas by Ehresmann, [5].
Definition 5b. Space with Cartan connection of type \((G, H)\) over \(M\) is a quadruple \(S = S(M) = (P(M, G), \Gamma, E = P[G/H], s)\) such that \(\dim M = \dim S\) and the absolute differential \(\nabla_{\Gamma} s\) is formed by regular 1-jets.

We deduce that both concepts are naturally equivalent. In the case b), \(s\) defines a reduction to subgroup \(H\)

\[ Q = \{ u \in P, \tilde{u}(c) = s(p(u)) \}, \]

where \(p: P \to M\) is the bundle projection. Write \(\omega\) for the restriction of the connection form \(\omega_{\Gamma}: TP \to g\) to \(Q\). Clearly, \(\omega: TQ \to g\) satisfies (ii) and (iii) from Definition 5a.

Lemma 1. \(\omega\) satisfies (i), iff the 1-jets \((\nabla_{\Gamma} s)(x)\) are regular for all \(x \in M\).

Proof. The vertical tangent bundle \(VE\) is an associated bundle \(P[T S]\) and \(T \tilde{u}: TS \to T E_x\) is the induced frame map on \(VE\). Consider \(X = \frac{d \gamma(0)}{dt} \in T_x M\), \(\gamma: \mathbb{R} \to M\). Then

\[ (T \tilde{u})^{-1}((\nabla_{\Gamma} s)(X)) = \frac{d}{dt} \bigg|_0 \tilde{\varphi}(\gamma(t))^{-1}(s(\gamma(t))). \]

Since \(G\) acts transitively on \(S\), we have

\[ \tilde{\varphi}(\gamma(t))^{-1}(s(\gamma(t))) = \delta(t)c, \quad \delta: \mathbb{R} \to G. \]

Write \(Z = \frac{d}{dt} \bigg|_0 \tilde{\varphi}(\gamma(t))\delta(t) \in T_u Q\). By the definition of the connection form, we have

\[ (T \tilde{u})^{-1}((\nabla_{\Gamma} s)(X)) = \omega(Z) + \mathfrak{h} \in T_c S. \]

These vectors are linearly independent for a basis of \(T_x M\), iff \(\omega(u)\) is a linear isomorphism. \(\square\)

Using Lemma 1 one easily verifies that Definitions 5a and 5b are equivalent. In what follows, \(S(M)\) will be called a Cartan space and \(\omega\) will be called its connection form.

Consider an \(n\)-submanifold \(N \subset M\). If we restrict all objects in question over \(N\), we obtain

\[ (P_N, \Gamma_N, E_N, s_N) = S_N = (Q_N, \omega_N). \]

Then we have the situation from Section 2. By induction we deduce that \(((k \nabla_{\Gamma_N}) s_N)(x)\) depends on \(k^r \tilde{N}\) only. Write \(S^r_n = (K^n_\ast S)_c\) and \(\tilde{S}^r_n = (\tilde{K}^n_\ast S)_c\). Clearly, both \(S^r_n\) and \(\tilde{S}^r_n\) are \(H\)-spaces.
**Absolute differentiation in Cartan spaces**

**Definition 6.** The map

$$
\Gamma^r_n: K^r_n M \to Q[S^r_n], \quad \Gamma^r_n(k^r_x N) = ((k\nabla \Gamma_n) s_N)(x)
$$

is called the formal absolute contact \((n, r)\)-differentiation on \(S\). The map

$$
\Gamma^r_N = \Gamma^r_n \circ k^r_N: N \to Q_N[S^r_n]
$$

is said to be the \(r\)-th absolute contact differential of \(N\).

If \(S\) is the Klein space \(G/H\) with the canonical flat connection, we have

$$
Q[S^r_n] = K^r_n S.
$$

Then, by Remark 1, \(\Gamma^r_n\) is the identity of \(K^r_n S\) composed with the injection \(K^r_n S \hookrightarrow \bar{K}^r_n S = Q[S^r_n]\).

**Remark 2.** In [13], our investigation of the \(r\)-th absolute contact differential of \(N\) was based on Ehresmann’s idea of higher order prolongations of connection \(\Gamma\). But (5) implies directly that both approaches coincide. So we can use our results from [13] in what follows.

## 4 Submanifolds of Klein spaces

If \(N\) is a submanifold of Klein space \(S\), we write \(G_N\) for the restriction of principal bundle \(G(S, H)\) over \(N\). Then \(k^r_N\) is a section \(N \to G_N[S^r_n]\), that is sometimes called the fundamental \(r\)-th order field on \(N\). It was pointed out by G. F. Laptev, [17] (but in a computational form only), that a modification of the Cartan method of moving frames leads to the coordinate functions of \(k^r_N\).

In general, consider a principal \(G\)-bundle \(p: P \to M\), a left \(G\)-space \(F\) and the associated bundle \(E = P[F] = P \times_G F\). Every \(u \in P_x\), \(x \in M\), is interpreted as the frame map \(\tilde{u}: F \to E_x\), \(\tilde{u}(a) = \{u, a\}\), \(a \in F\). For every section \(s: M \to E\), the induced map

$$
P \to F, \quad u \mapsto \tilde{u}^{-1}(s(p(u)))
$$

is said to be the frame form of \(s\), [16]. If \(z^a\) are some local coordinates on \(F\), then the locally defined compositions of (7) with \(z^a\) are called the coordinate functions of \(s\).

Further, every left action \(l: G \times F \to F\) induces the infinitesimal action \(\lambda: g \times F \to TF\). The Maurer-Cartan form \(\varphi: TG \to g\) yields an identification \(G \times g \approx TG\). This defines an involutive distribution \(\Lambda\) on \(G \times F\),

$$
\Lambda(g, z) = \{(g, X), \lambda(X, z)\}; X \in g\}, \quad g \in G, z \in F,
$$
whose integral manifolds determine action \( l \). If we consider some local coordinates \( z^a \) on \( F \) and a basis of \( \mathfrak{g} \), the coordinate expression of \( \lambda \) is of the form

\[
dz^a = \eta_I^a(z)\xi^I, \quad (\xi^I) \in \mathfrak{g}, \quad I = 1, \ldots, \dim G.
\]

Then the equations of \( \Lambda \) are

\[
dz^a - \eta_I^a(z)\varphi^I = 0, \quad \varphi = (\varphi^I).
\]

They are usually called the equations of the infinitesimal action \( \lambda \) of \( G \) on \( F \).

The elements of \( G_N \) are said to be zero order frames of \( N \). They are characterized by the property that the image \( \tilde{g}(c) \) of \( c \in S \) under the frame map \( \tilde{g} \) lies in \( N \). So the frame form of \( k_N^r \) is a map \( G_N \to S_n^r \). Consider the canonical coordinates

\[
x^i, x^p, \quad i = 1, \ldots, n, \quad p = n + 1, \ldots, m
\]

on \( \mathbb{R}^{n,m-n} \). The induced coordinates on the \( r \)-th jet prolongation \( J^r\mathbb{R}^{n,m-n} \) are

\[
x^p, x^p_i, \ldots, x^p_{i_1 \cdots i_r}.
\]

Choose a local coordinate system \( x^i, x^p \) on \( S \) centered at \( c \). This identifies locally \( S^r_n \) with \( J^r_0\mathbb{R}^{n,m-n} \). We write

\[
(8) \quad (a^p_i, a^p_{ij}, \ldots, a^p_{i_1 \cdots i_r}): G_N \to S^r_n
\]

for the locally defined coordinate functions of section \( k_N^r \) of \( G_N[S^r_n] \).

The algorithm for finding (8) by the Cartan-like procedure from [17] is described in [14]. This general approach is based on the use of zero order frames of \( N \). However, the evaluations in zero order frames are top-heavy because of the nontrivial topological character of the classical Grassmann manifolds. Thus, in practice one always uses the first order frames of \( N \). So, also here we restrict ourselves to the first order frames.

Assume that \( H \) acts transitively on \( S^1_n \), which is satisfied for all classical Klein spaces. Choose a point \( c_n \in S^1_n \), write \( H_1 \) for its stability group and \( S^r_{n1} \) for the fiber of \( S^r_n \to S^1_n \) over \( c_n \). Clearly, \( S^r_{n1} \) is an \( H_1 \)-space. A frame \( \tilde{g} \in G_N \) is said to be first order frame of \( N \), if \( \tilde{g}(c_n) = T_{2c}N \). Clearly, the space \( G_{N1} \) of all first order frames of \( N \) is a principal bundle \( G_{N1}(N, H_1) \). If we restrict ourselves to the first order frames, the frame form of \( k_N^r \) is a
Absolute differentiation in Cartan spaces

map $G_{N1} \to S_{n1}^r$. Assume further that the equations of $c_n$ are $dx^p = 0$. In
other words, the jet coordinates of $c_n$ are $x^p_i = 0$. So the first order frames
of $N$ are characterized by $a_i^p = 0$. The induced global coordinates on $S_{n1}^r$
are $x^p_{ij}, \ldots, x^p_{i_1 \ldots i_r}$. If we interpret $k^*_N$ as a section of the associated bundle
$G_{N1}[S_{n1}^r]$, then its coordinate functions

$$(a^p_{ij}, \ldots, a^p_{i_1 \ldots i_r}) : G_{N1} \to S_{n1}^r$$

are globally defined.

The simplest algorithm appears in the case there exists an Abelian subgroup $K \subset G$ such that $\mathfrak{g}$ is the product $\mathfrak{k} \times \mathfrak{h}$. (But all classical Klein spaces have this property. For example, if $A_m$ is an $m$-dimensional affine space, we have $G = GA(m)$, $H = GL(m)$ and $K = \mathbb{R}^m \subset GA(m)$ is the Abelian
subgroup of all translations on $A_m$.) We choose a basis of $\mathfrak{g}$

$$e_\alpha, e_\lambda, \quad \alpha, \beta = 1, \ldots, m, \quad \lambda, \mu, \nu = m + 1, \ldots, \dim G$$

such that $e_\lambda$ lie in $\mathfrak{h}$ and $e_\alpha$ is a basis of $\mathfrak{k}$.

This assumption is equivalent to the following relations on the structure
constants of $G$

$$(9) \quad c^\alpha_{\beta \gamma} = 0, \quad c^\lambda_{\alpha \beta} = 0, \quad c^\alpha_{\lambda \mu} = 0.$$  

Hence the coordinate form of the structure equations $d\varphi + \frac{1}{2} [\varphi, \varphi] = 0$ of $\varphi$
is

$$(10) \quad d\varphi^\alpha = c^\alpha_{\lambda \beta} \varphi^\beta \wedge \varphi^\lambda, \quad d\varphi^\lambda = c^\lambda_{\alpha \mu} \varphi^\mu \wedge \varphi^\alpha - \frac{1}{2} c^\lambda_{\mu \nu} \varphi^\mu \wedge \varphi^\nu.$$  

We shall write $\pi^\lambda$ for the restriction of $\varphi^\lambda$ to $H$. The bundle projection
$G \to S$ identifies locally $K$ with $S$. So the basis $e_\alpha$ defines local coordinates
$x^\alpha$ on $S$, with $(x^\alpha) = (x^i, x^p)$.

In what follows we shall write $\varphi$ for the restriction $\varphi_{N1}$ of $\varphi$ to $G_{N1}$, as
usual in concrete investigations. So our starting point are the equations

$$\varphi^p = 0.$$  

If we substitute them into (10), we obtain

$$0 = c^p_{i \lambda} \varphi^\lambda \wedge \varphi^i.$$
Using the Cartan lemma, we find

\[ c^p_{i\lambda} \varphi^\lambda = a^p_{ij} \varphi^j, \quad a^p_{ij} = a^p_{ji}, \]

where \( a^p_{ij} \) are some functions on \( G_{N1} \).

In [14], we deduced

**Proposition 3.** \( a^p_{ij} \) coincide with the coordinate functions of \( k^2_N \) on \( G_{N1} \).

In particular, (11) implies that the differential equations of \( H_1 \) are

\[ c^p_{i\lambda} \pi^\lambda = 0. \]

Now we apply exterior differentiation to (11). Using the structure equations, we obtain an expression of the form

\[ [da^p_{ij} - \Phi^p_{ij\lambda}(a^q_{kl}) \varphi^\lambda] \wedge \varphi^j = 0. \]

If we apply Cartan lemma to (12), we obtain

\[ da^p_{ij} - \Phi^p_{ij\lambda}(a^q_{kl}) \varphi^\lambda = a^p_{ijk} \varphi^k. \]

(We shall see that \( a^p_{ijk} \) are the additional coordinate functions of \( k^3_N \) on \( G_{N1} \).)

This procedure can be iterated. Assume by induction that after \( r-3 \) steps we have the equations of the infinitesimal action of \( H_1 \) on \( S^r_{n1} \)

\[ dx^p_{ij} - \Phi^p_{ij\lambda}(x^q_{kl}) \pi^\lambda = 0, \]

\[ \vdots \]

\[ dx^p_{1i...i_{r-2}} - \Phi^p_{1i...i_{r-2}\lambda}(x^q_{kl}, \ldots, x^q_{i_{r-3}j_{r-2}}) \pi^\lambda = 0 \]

with \( c^p_{i\lambda} \pi^\lambda = 0 \), and it holds

\[ c^p_{i\lambda} \varphi^\lambda = a^p_{ij} \varphi^j; \]

\[ \vdots \]

\[ da^p_{1i...i_{r-2}} - \Phi^p_{1i...i_{r-2}\lambda}(a^q_{j1j2}, \ldots, a^q_{j1...j_{r-2}}) \varphi^\lambda = a^p_{1i...i_{r-2}j} \varphi^j. \]

If we apply exterior differentiation to the last row and use all these equations, we obtain certain relations of the form

\[ [da^p_{1i...i_{r-2}k} - \Phi^p_{1i...i_{r-2}k\lambda}(a^q_{j1j2}, \ldots, a^q_{j1...j_{r-1}}) \varphi^\lambda] \wedge \varphi^k = 0. \]

In [14], we deduced
Proposition 4. The additional equations of the infinitesimal action of $H_1$ on $S_{n1}^{r-1}$ are

$$dx_{i_1...i_{r-1}}^p - \Phi_{i_1...i_{r-1}\lambda}(x_{j_1j_2},\ldots,x_{j_1...j_{r-1}}^a)p^\lambda = 0$$

with $c_{i_1...i_{r-1}\lambda}^p = 0$.

The additional coordinate functions $a_{i_1...i_r}^p$ of $k_N^r$ on $G_{N1}$ satisfy

$$da_{i_1...i_{r-1}}^p - \Phi_{i_1...i_{r-1}\lambda}(a_{j_1j_2}^a,\ldots,a_{j_1...j_{r-1}}^a)p^\lambda = a_{i_1...i_r}^p \varphi^\lambda.$$

The Cartan method of moving frames is usually used for finding differential invariants of $N \subset S$ and for solving the equivalence problem for $N$. The fact that the above procedure yields the equations of the infinitesimal action of $H_1$ on $S_{n1}^r$ was used in [17] for local computations of the geometric objects of $N$. Our analysis of these algorithms led us to the following conceptual definition, [14]. Let $A$ be an $H$-space.

Definition 7. A geometric $(n, r)$-object on $S$ is an $H$-equivariant map $\mu: S_n^r \to A$.

Since $\mu$ is an $H$-map, it induces the associated bundle morphism $\bar{\mu}: K_n^r S \to G[A]$. The map

$$\mu_N = \bar{\mu} \circ k_N^r: N \to G_N[A]$$

is called the value of geometric $(n, r)$-object $\mu$ on $N$. More generally, let $W \subset S_n^r$ be an $H$-invariant submanifold. An $n$-submanifold $N \subset S$ is said to be of type $W$, if the values of $k_N^r$ lie in $G_N[W]$. We can introduce a geometric object of type $W$ as an $H$-map $\mu: W \to A$. For a submanifold $N$ of type $W$, $\bar{\mu} \circ k_N^r$ is the value of $\mu$ on $N$. Very simple examples of $W$ are elliptic, parabolic and hyperbolic contact $(2, 2)$-elements on Euclidean 3-space.

The equations of the infinitesimal action can be used, at least locally, for constructing the equivariant maps. A general global result is due to R. Palais, [21]. We refer the reader to [14] for more details concerning the case of contact elements. We underline that the globality of the infinitesimally equivariant maps frequently follows from the geometrical interpretation of the results of evaluations.

In practice, one constructs the geometric objects of $N$ by using the first order frames. If we interpret $H$ as a principal $H_1$-bundle $H(H/H_1, H_1)$, then $S_n^r$ coincides with the associated bundle $H[S_{n1}^r]$. The left action of $H$ on $S_n^r$ has the form

$$\bar{h}\{h, y\} = \{\bar{h}h, y\}, \quad h, \bar{h} \in H, y \in S_{n1}^r.$$
Let $B$ be an $H_1$-space. The associated bundle $H[B]$ is an $H$-space with respect to the action

$$\tilde{h}\{h, z\} = \{hh, z\}, \quad h, \tilde{h} \in H, z \in B.$$ 

This definition is correct, for $\tilde{h}\{hh_1, h_1^{-1}z\} = \{\tilde{hh}_1, h_1^{-1}z\} = \{\tilde{h}, z\}$, $h_1 \in H_1$. For every $H_1$-map $\nu: S_{n1}^r \to B$, the induced map $\tilde{\nu}: H[S_{n1}^r] \to H[B]$ is $H$-equivariant. So every $H_1$-map $\nu: S_{n1}^r \to B$ gives rise to a geometric $(n, r)$-object on $S$.

We underline that the concept of $r$-th order geometric object for $n$-submanifolds of $S$ is of universal character. Its specification to an $n$-submanifold $N \subset S$ (or to a submanifold of type $W$) is constructed by means of the contact elements, so that it is independent of parametrizations of $N$.

The differential invariants of submanifolds are the simplest example of geometric objects. In this case, $A = \mathbb{R}$ with the identity action of $H$. Further, if we consider the action of $H$ on $\mathbb{R}$ by means of homotheties, we obtain the so-called relative invariants.

**Remark 3.** According to the Cartan-like algorithm of this section (see [14] for the use of zero order frames), the geometric objects of a submanifold $N \subset S$ are determined by the restriction $\varphi_N$ of the Maurer-Cartan form of $G$ over $N$. This corresponds to the well known role of $\varphi_N$ in the equivalence problem for $N$, see [3], [7]. We recall that this role is based on the fact that, for a connected manifold $N$, two maps $f_1, f_2: N \to G$ are congruent, i.e. there exists $g \in G$ such that $f_1(x) = gf_2(x)$ for all $x \in N$, if and only if $\varphi \circ Tf_1 = \varphi \circ Tf_2: TN \to \mathfrak{g}$.

## 5 Submanifolds of Cartan spaces

Consider a Cartan space $S(M)$ such that $H$ acts transitively on $S_n^1$. Let $N \subset M$ be an $n$-submanifold. The elements of $Q_N$ are zero order frames of $N$, they are characterized by $\tilde{u}(c) \in s_N(N)$. A frame $u \in (Q_N)_x$ is said to be first order frame of $N$, if $\tilde{u}(c_n) = \Gamma^1_N(x)$. Analogously to Section 4, these frames form a reduction $Q_{N1}$ of $Q_N$ to $H_1$. In this situation, a frame $u \in Q_N$ is a first order frame of $N$, iff $\omega^h_N(u) = 0$. We write $\tilde{S}_n^r = (\tilde{K}_n^rS)_c$. Then $\tilde{K}_n^rS$ is an associated bundle $G[\tilde{S}_n^r]$. Further, we write $\tilde{S}_{n1}^r$ for the fiber $\tilde{S}_n^r \to S_n^1$ over $c_n$. 


The $r$-th absolute contact differential $\Gamma^r_N$ can be viewed as a section of the associated bundle $Q_{N1}[\bar{S}^r_{n1}]$. By (4), the coordinates $x^i$, $x^p$ identify $\bar{S}^r_n$ locally with $J^n_0 \mathbb{R}^{n,m-n}$. Hence the induced coordinates on $\bar{S}^r_{n1}$ are

$$x^p_{ij},\ldots,x^p_{i_1\ldots i_r}$$

arbitrary in all subscripts. The coordinate functions of $\Gamma^r_N$

$$(b^p_{ij},\ldots,b^p_{i_1\ldots i_r}): Q_{N1} \to \bar{S}^r_{n1}$$

are globally defined.

Assume the existence of $K \subset G$ as in Section 4. Then we have the following simple procedure for finding the coordinate functions of $\Gamma^r_N$, in which the role of the Maurer-Cartan form $\varphi$ from Section 4 is replaced by the connection form $\omega$. We write $\omega$ for the restriction $\omega_{N1}$ of $\omega$ to $Q_{N1}$. So our starting point are the equations

$$\omega^p = 0.$$

In [13], we deduced

**Proposition 5.** We have

$$c^p_{i\lambda} \omega^\lambda = b^p_{ij} \omega^j.$$  

The algorithm from Section 4 is now modified as follows, [13]. Write formally the relations, with arbitrary $x^p_{ij}$,

$$c^p_{i\lambda} \varphi^\lambda = x^p_{ij} \varphi^j.$$  

Applying exterior differentiation to (14), using the structure equations of $\varphi$ and $\varphi^p = 0$, we find an expression of the form

$$[dx^p_{ij} - \Psi^p_{ij\lambda}(x^q_{kl})\varphi^\lambda] \wedge \varphi^j = 0.$$  

**Proposition 6.** The equations of the infinitesimal action of $H_1$ on $\bar{S}^2_{n1}$ are

$$dx^p_{ij} - \Psi^p_{ij\lambda}(x^q_{kl})\pi^\lambda = 0 \quad \text{with} \quad c^p_{i\lambda} \pi^\lambda = 0.$$
This procedure can be iterated. Assume that after \( r-3 \) steps we have deduced the equations of the infinitesimal action of \( H_1 \) on \( \bar{S}_{n1}^{r-2} \)

\[
dx_{ij}^p - \Psi_{ij\lambda}^p(x_{kl}^q)\pi^\lambda = 0,
\]

(15)

\[
dx_{i_1 \ldots i_{r-2}}^p - \Psi_{i_1 \ldots i_{r-2}\lambda}^p(x_{j_1}^{q_1}, \ldots, x_{j_{r-1}}^{q_{r-1}})\pi^\lambda = 0
\]

with \( c_{i\lambda}^p \pi^\lambda = 0 \). Then we write formally the relations, with arbitrary \( x^p_{i_1 \ldots i_{r-2} j} \),

\[
dx_{i_1 \ldots i_{r-2} j}^p - \Psi_{i_1 \ldots i_{r-2}\lambda}^p(x_{j_1}^{q_1}, \ldots, x_{j_{r-1}}^{q_{r-1}})\varphi^\lambda = x^p_{i_1 \ldots i_{r-2} j} \varphi^j.
\]

Applying exterior differentiation to (16) with \( \varphi^p = 0 \), using the structure equations of \( \varphi \) and (15), we obtain an expression of the form

\[
[dx_{i_1 \ldots i_{r-2} j}^p - \Psi_{i_1 \ldots i_{r-2}\lambda}^p(x_{j_1}^{q_1}, \ldots, x_{j_{r-1}}^{q_{r-1}})\varphi^\lambda] \wedge \varphi^j = 0.
\]

**Proposition 7.** The additional equations of the infinitesimal action of \( H_1 \) on \( \bar{S}_{n1}^{r-2} \) are

\[
dx_{i_1 \ldots i_{r-1}}^p - \Psi_{i_1 \ldots i_{r-1}\lambda}^p(x_{j_1}^{q_1}, \ldots, x_{j_{r-1}}^{q_{r-1}})\pi^\lambda = 0 \quad \text{with} \quad c_{i\lambda}^p \pi^\lambda = 0.
\]

The coordinate functions \( b^p_{ij}, \ldots, b^p_{i_1 \ldots i_{r-2} j} \) of \( \Gamma^r_N \) on \( Q_{N1} \) satisfy (13) and

\[
 db_{ij}^p - \Psi_{ij\lambda}^p(b_{kl}^q)\omega^\lambda = b_{ijk}^p\omega^k,
\]

\[
 \vdots
\]

\[
 db_{i_1 \ldots i_{r-1}}^p - \Psi_{i_1 \ldots i_{r-1}\lambda}^p(b_{j_1}^{q_1}, \ldots, b_{j_{r-1}}^{q_{r-1}})\omega^\lambda = b_{i_1 \ldots i_{r-1}j}^p \omega^j.
\]

**Remark 4.** We underline that the absolute contact differential of any order of \( N \) is determined by the restriction \( \omega_N \) of the connection form \( \omega \) over \( N \). This is an important analogy of Remark 3. Clearly, Section 4 can be viewed as a special case, provided we consider \( S \) as a flat Cartan space.

Now we generalize the concept of geometric \((n, r)\)-object to Cartan spaces. Let \( A \) be an \( H \)-space.

**Definition 8.** A geometric \((n, r)\)-object on \( S(M) \) is an \( H \)-equivariant map \( \mu: \bar{S}_n^r \to A \).
We also say that $\mu$ is a semiholonomic $(n, r)$-object. For $n = 1$ we have $\bar{S}_1^r = S_1^r$, so that there exist holonomic $(1, r)$-objects only.

So we have the induced bundle morphism $\bar{\mu}: Q[\bar{S}_n^r] \to Q[A]$. For a submanifold $N \subset M$, the composition

$$\mu_N = \bar{\mu} \circ \Gamma^r_N: N \to Q_N[A]$$

is called the value of $\mu$ on $N$. More generally, if $W \subset \bar{S}_n^r$ is an $H$-invariant submanifold, then the $(n, r)$-objects of type $W$ are defined analogously to Section 4. Clearly, one can restrict himself to the first order frames of $N$ in the same way as above.

### 6 The torsion-free case

For a Cartan space $S(M)$, Sharpe defines its curvature $\Omega$ by

$$d\omega + \frac{1}{2} [\omega, \omega] = \Omega,$$

[22]. So $\Omega$ is the restriction of the curvature $\Omega_{\Gamma}$ of $\Gamma$ to $Q$. It is well known that $\Omega_{\Gamma}$ can be interpreted as a map

$$\Omega_{\Gamma}: P \times_M \bigwedge^2 TM \to g.$$

Hence we may consider $\Omega$ as a map

$$\Omega: Q \times_M \bigwedge^2 TM \to g.$$

The coordinate form of (17) is

$$d\omega^I + \frac{1}{2} c^I_{JK} \omega^J \wedge \omega^K = R^I_{\alpha\beta} \omega^\alpha \wedge \omega^\beta, \quad I, J, K = 1, \ldots, \dim G.$$

In [9] we introduced the following concept in a slightly different, but equivalent way. Write $L = g/h = T_eS$ and $\psi: g \to L$ for the factor projection.

**Definition 9.** The composition $\sigma = \psi \circ \Omega: Q \times_M \bigwedge^2 TM \to L$ is called the torsion of $S$. 
The absolute differentiation with respect to $\Gamma$ identifies $T_x M$ with $T_{s(x)} E_x$. Clearly, $L$ is an $H$-space and the corresponding associated bundle satisfies
\begin{equation}
Q[L] \approx \bigcup_{x \in M} T_{s(x)} E_x.
\end{equation}
Hence $\sigma$ can be interpreted as a section
\begin{equation}
\sigma: M \to Q[L \otimes \wedge^2 L^*].
\end{equation}
By (19), the coordinate expression of $\sigma$ is
\begin{equation}
R_{\alpha\beta}^\gamma \omega^\alpha \wedge \omega^\beta.
\end{equation}
This implies that $\sigma$ coincides with the standard torsion in the classical case of an affine connection on the linear frame bundle of $M$.

**Remark 5.** The concept of higher order torsions of Cartan spaces is discussed from a similar point of view in [10].

Our result from [9] can be now formulated as follows. (Another approach to this assertion will be discussed in Section 8.)

**Proposition 8.** If the torsion of $S$ vanishes, then the values of $\Gamma_n^2$ are holonomic contact $(n,2)$-elements.

In particular, this is true in the case of a Riemannian manifold $(M, g)$, that is considered as a Cartan space $E_m(M)$ with respect to the Levi-Civita connection. Thus, from the viewpoint of our approach, the second-order geometric objects on submanifolds of Riemannian spaces are of the same type as in the case of submanifolds of Euclidean spaces.

## 7 Universal tensor bundles for submanifolds

We present another situation, in which the idea of universal geometric object for submanifolds plays a remarkable role. We start with the case of an arbitrary manifold $M$. The vertical bundle $VN$ of $N \subset M$ is the factor bundle $(TM)_N/TN$. A section

\begin{equation}
N \to \bigotimes^a TN \otimes \bigotimes^b VN \otimes \bigotimes^c T^*N \otimes \bigotimes^d V^*N
\end{equation}
will be called a tangent-vertical tensor field on $N$.

For every $\xi \in (K^1_n M)_x$, we denote by $\tau(\xi) \subset T_x M$ the corresponding $n$-dimensional subspace and by $\nu(\xi) = T_x M/\tau(\xi)$ the vertical space. Then

$$H^1_n M = \bigcup_{\xi \in K^1_n M} \tau(\xi) \quad \text{and} \quad V^1_n M = \bigcup_{\xi \in K^1_n M} \nu(\xi)$$

are vector bundles over $K^1_n M$ and we have an exact sequence (see also [19])

$$(23) \quad 0 \rightarrow H^n_1 M \rightarrow (\pi^1_0)^* TM \rightarrow V^n_1 M \rightarrow 0.$$ 

**Definition 10.** The induced bundle over $K^r_n M$

$$H^r_n M = (\pi^r_1)^* H^1_n M \quad \text{or} \quad V^r_n M = (\pi^r_1)^* V^1_n M$$

is called the universal horizontal or vertical $(n, r)$-bundle over $M$, respectively.

We define

$$I^{r,a,b,c,d}_{n,c,d} = a \otimes H^a_n M \otimes b \otimes V^b_n M \otimes c \otimes H^c_1 M \otimes d \otimes V^d_1 M.$$

Every section $\varphi: K^r_n M \rightarrow I^{r,a,b,c,d}_{n,c,d}$ determines a tangent-vertical tensor field $\varphi \circ k^r_N$ on every $n$-submanifold $N$.

For example, consider the $m$-dimensional affine space $A_m$ and $N \subset A_m$. For a vector $X \in T_x N$, we define $\varphi_x(X) \in V_x N$ as follows. Take a curve $\gamma(t)$ on $N$ such that $\frac{d\gamma(t)}{dt} = X$. In the case of $A_m$, the acceleration $\frac{d^2\gamma(t)}{dt^2}$ belongs to $T_x A_m$. Its projection into $V_x N$ depends on $\frac{d\gamma(t)}{dt}$ only. This defines a map $\varphi_x: T_x N \rightarrow V_x N$, that generates a quadratic $V N$-valued form $\varphi_N$ on $N$. Its universal version is a section

$$(25) \quad \varphi: K^2_n A_m \rightarrow V^2_n A_m \otimes S^2(H^2_1 A_m).$$

**Definition 11.** We say that $\varphi$ is the universal fundamental form for $n$-submanifolds of $A_m$.

For every submanifold $N \subset A_m$, $\varphi \circ k^2_N$ is the fundamental form of $N$.

We remark that an application of the concept of universal tensor bundles to the calculus of variations on submanifolds is presented in [18]. Another interesting application of this concept can be found in [6].

In the semiholonomic case, we construct the pullbacks

$$\bar{H}^r_n M \rightarrow \bar{K}^r_n M \quad \text{and} \quad \bar{V}^r_n M \rightarrow \bar{K}^r_n M$$

in the same way as in Definition 10.
8 The reduced torsion and the difference tensor

Consider a submanifold $N$ of a Cartan space $S(M)$. In the tangent space $T_{s(x)}E_x$, we have an $n$-dimensional subspace $\tau^N_{1}(x)$ corresponding to $\Gamma^1_N(x)$. The factor space

$$\nu^N_N(x) = T_{s(x)}E_x/\tau^N_N(x)$$

will be called the vertical space of $N$ at $x$. Write $\sigma_N$ for the restriction of $\sigma$ to $Q_N$.

**Definition 12.** The projection $\tilde{\sigma}_N(x)$ of $\sigma_N(x)$ into $\nu^N_N(x)$ is called the reduced torsion of $N$ at $x$.

The universal version of the reduced torsion is closely related with the general concept of difference tensor of semiholonomic contact $(n,2)$-elements. According to [9], every semiholonomic 2-jet $X \in \bar{J}^2_x(M,N)_y$ determines the difference tensor $\Delta X \in T_yN \otimes \wedge^2 T^*_xM$. If $(x^i, y^\alpha, y^\alpha_i, y^\alpha_{ij})$ are the canonical coordinates on $J^2(\mathbb{R}^n, \mathbb{R}^m)$, then the coordinate expression of $\Delta X$ is

$$\Delta X = (x^i, y^\alpha, y^\alpha_{[ij]}).$$

So $\Delta X = 0$, iff $X$ is a holonomic 2-jet.

Consider $X \in \text{reg } T^2_nM$ and $\xi = k(X) \in \bar{K}^2_nM$. The underlying 1-jet $\pi_1^2X \in \text{reg } T^1_nX$ identifies $\mathbb{R}^n$ with $\tau(\xi_1)$, $\xi_1 = \pi_1^2\xi$. The projection of $\Delta X$ into $\nu(\xi_1)$ depends on $\xi$ only. This defines

$$\delta(\xi) \in \nu(\xi_1) \otimes \wedge^2 \tau(\xi_1)^*,$$

that will be called the difference tensor of $\xi$. Hence $\delta$ is a section

$$\delta: \bar{K}^2_nM \to \bar{V}^2_nM \otimes \wedge^2 \bar{H}^2_nM$$

that is said to be the contact difference tensor. Under the identification (4) of $\tau\bar{K}^2_n\mathbb{R}^m$ with $\bar{J}^2\mathbb{R}^{n,m-n}$, both approaches to the difference tensor coincide. Clearly, $\xi \in K^2_nM$ iff $\delta(\xi) = 0$. 

If we analyze $\tilde{\sigma}$ from the viewpoint of semiholonomic $(n, 2)$-objects, we realize that it is determined by an $H$-map

$$\bar{S}_n^2 \to (V_n^{-1}S)_c \otimes \wedge^2 (H_n^{1*}S)_c,$$

that coincides with the contact difference tensor. This yields

**Proposition 9.** We have $\tilde{\sigma}_N(x) = 0$ iff $\Gamma^2_1(x)$ is holonomic.

Hence Proposition 8 is a direct consequence of Proposition 9.

To illustrate the use of the algorithm from Section 5, we rederive this assertion by direct evaluation under the additional assumptions on $S_n^1$ and $K$. In the first order frames of $N$, (19) and (22) imply

$$0 = c^p_{ij} \omega^i \wedge \omega^j + R^p_{ij} \omega^i \wedge \omega^j,$$

where $R^p_{ij} \omega^i \wedge \omega^j$ is the coordinate expression of $\tilde{\sigma}_N$. The coordinate functions of $\Gamma^2_N$ satisfy $c^p_{ij} \omega^i = b^p_{ij} \omega^j$. Hence $\tilde{\sigma}_N = 0$ is equivalent to $b^p_{ij} \omega^i \wedge \omega^j = 0$, i.e. $b^p_{ij} = b^p_{ji}$.

As a concrete example, we consider a 2-submanifold $N_2 \subset P_3$ of a 3-space with projective connection. The projective 3-space $P_3$ is generated by an affine 4-space $A_4$ and we write $\{u\} \in P_3$ for the point determined by a nonzero vector $u \in A_4$. We fix a basis $u_0, u_1, u_2, u_3$ of $A_4$ and define $c^1 = \{u_0\}$ and $c^2$ as the linear space in $T_cP_3$ corresponding the 2-plane determined by $\{u_0\}, \{u_1\}, \{u_2\}$. The Maurer–Cartan form of the projective group $GP(3)$ is $(\varphi^a_b)$ with $\varphi^a_a = 0$, $a, b = 0, 1, 2, 3$, and we have

$$d\varphi^b_a = \varphi^c_a \wedge \varphi^b_c \quad \text{with} \quad \varphi^a_a = 0.$$

The differential equations of $H$ are $\varphi^1_0 = \varphi^2_0 = \varphi^3_0 = 0$. One verifies directly that condition (9) is satisfied. Then the relation

$$d\varphi^3_0 = \varphi^0_0 \wedge \varphi^3_0 + \varphi^1_0 \wedge \varphi^3_1 + \varphi^2_0 \wedge \varphi^3_2 + \varphi^3_0 \wedge \varphi^3_3$$

implies that the additional differential equations of $H_1$ are $\pi^1_1 = 0$, $\pi^3_3 = 0$.

The restriction $(\omega^a_0), \omega^a_a = 0$ of the connection form $\omega$ of $P_3$ to the first order frames of $N_2$ is characterized by $\omega^3_0 = 0$. If we write $\omega^1_0 = \omega^1, \omega^2_0 = \omega^2$, then (13) yields

$$\omega^3_1 = b_{11} \omega^1 + b_{12} \omega^2, \quad \omega^3_2 = b_{21} \omega^1 + b_{22} \omega^2.$$
Then (30) is of the form

\[
0 = \omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^3 + 2R^3_0 \omega^1 \wedge \omega^2.
\]

Hence (33) implies \(2R^3_0 = b_{21} - b_{12}\). So the equations (14) are of the form

\[
\varphi^3_1 = x_{11}\varphi^1_0 + x_{12}\varphi^2_0, \quad \varphi^3_2 = x_{21}\varphi^1_0 + x_{22}\varphi^2_0.
\]

Applying the procedure from Section 5, we obtain the equations of the infinitesimal action of \(H_1\) on \(\bar{S}_{21}\)

\[
\begin{align*}
&dx_{11} + x_{11}(\pi^0_0 - 2\pi^1_1 + \pi^3_1) - x_{12}\pi^2_1 - x_{21}\pi^1_1 = 0, \\
&dx_{12} + x_{12}(\pi^0_0 - \pi^1_1 - \pi^2_2 + \pi^3_3) - x_{11}\pi^2_1 - x_{22}\pi^1_1 = 0, \\
&dx_{21} + x_{21}(\pi^0_0 - \pi^1_1 - \pi^2_2 + \pi^3_3) - x_{11}\pi^2_1 - x_{22}\pi^1_1 = 0, \\
&dx_{22} + x_{22}(\pi^0_0 - 2\pi^2_1 + \pi^3_3) - x_{12}\pi^1_2 - x_{21}\pi^2_2 = 0.
\end{align*}
\]

In particular,

\[
\begin{align*}
&(x_{21} - x_{12}) + (x_{21} - x_{12})(\pi^0_0 - \pi^1_1 - \pi^2_2 + \pi^3_3) = 0,
\end{align*}
\]

so that \(R^3_0\) is a relative invariant. Further, in the non-parabolic case \(x_{12}x_{21} - x_{11}x_{22} \neq 0\) we find

\[
d((x_{21} - x_{12})^2/(x_{12}x_{21} - x_{11}x_{22})) = 0.
\]

Hence \((R^3_0)^2/(b_{12}b_{21} - b_{11}b_{22})\) is an absolute invariant. Its geometric interpretation was found already by É. Cartan [4].

**Remark 6.** There exists a natural symmetrization \(\text{Sym} : \bar{J}^2(M,N) \to J^2(M,N)\) of semiholonomic 2-jets, [11]. In coordinates, one verifies directly that \(\text{Sym}\) preserves the jet composition

\[
\text{Sym}(Y \circ X) = \text{Sym}(Y) \circ \text{Sym}(X), \quad X \in \bar{J}^2_y(M_1, M_2) , \quad Y \in \bar{J}^2_y(M_2, M_3)z.
\]

Hence (3) implies that there is an induced symmetrization of contact \((n,2)\)-elements

\[
\text{Sym} : K^2_n M \to K^2_n M.
\]

We remark that in some geometric constructions \(\Gamma^N_N(x)\) enters via its symmetrization \(\text{Sym}(\Gamma^N_N(x))\). But this is not the case of the preceding example.

**Remark 7.** Some general aspects of the holonomicity problem for \(\Gamma^r_N\) in the case \(r > 2\) are studied in [12]. The case of \(N_2 \subset P_3\) is treated geometrically in [8].
9 Induced bundles over submanifolds

We point out that the idea of universality can be applied to a wide class of \( r \)-th order geometric objects over submanifolds. We write \( \text{reg} T^r_n M = P^r_n M \) if we consider it as a principal bundle over \( K^r_n M \) with structure group \( G^r_n \). For every \( n \)-submanifold \( N \subset M \), the map \( k^r_N \) induces a bundle \( (k^r_N)^* P^r_n M \to N \) that coincides with the \( r \)-th order frame bundle \( P^r N \). Let \( B \) be a \( G^r_n \)-space.

**Definition 13.** The associated bundle \( P^r_n M[B] \to K^r_n M \) is called the universal \( B \)-bundle of type \((n,r)\) over \( M \).

The map \( k^r_N \) induces the associated bundle \( (k^r_N)^* (P^r_n M[B]) \approx P^r N[B] \).

For \( q > r \), we can construct the pullback \( (\pi^q_r)^* P^r_n M = P^{r,q}_n M \to \) \( K^q_n M \) as well. In general, consider a principal bundle \( P(Z,K) \) and a left action of \( G \) on \( P \) commuting with the right action of \( K \) on \( P \), i.e.

\[
g(uk) = (gu)k, \quad g \in G, \; u \in P, \; k \in K.
\]

If \( B \) is a left \( K \)-space, we have an induced left action of \( G \) on \( P[B] \),

\[
g\{u, b\} = \{gu, b\}.
\]

This is a correct definition, for

\[
g\{uk, k^{-1}b\} = \{g(uk), k^{-1}b\} = \{(gu)k, k^{-1}b\} = \{gu, b\}.
\]

For example, if \( r = 1 \) and \( B = \mathbb{R}^n \) with the standard action of \( G^1 \), then \( P^{1,q}_n M[\mathbb{R}^n] = H^q_\mathbb{R} M \). On the other hand, the vertical bundle \( V^q_\mathbb{R} M \) is not of this type.

In the case of \( P^r_n S(K^r_n S, G^r_n) \) and a \( G^r_n \)-space \( B \), we obtain a left action of \( G \) on \( P^r_n S[B] \). Denote by \( \tilde{B} = (P^r_n S[B])_c \) the fiber over \( c \in S \). Hence our construction yields a left action of \( H \) on \( \tilde{B} \). Conversely, given a left action of \( H \) on \( \tilde{B} \), we have an identification \( G[B] \approx P^r_n S[B] \),

\[
\{g, \{u, b\}\} \mapsto \{gu, b\}.
\]
This is a correct definition, for
\[ \{gh, h^{-1}\{uk, k^{-1}b\}\} = \{gh, \{h^{-1}uk, k^{-1}b\}\} = \{gh, \{h^{-1}u, b\}\} = \{gu, b\}. \]

Consider another \(H\)-space \(A\),

**Definition 14.** An \(H\)-map \(\mu: \tilde{B} \to A\) is called geometric \(B\)-object of type \((n, r)\) over \(S\).

Since \(G[\tilde{B}] = P^r_nS[B]\), the map \(k_N^r: N \to K^r_nS\) induces
\[ \mu_N: P^rN[B] = G_N[B] \to G_N[A], \]
\[ \{u, b\} = \{g, \{g^{-1}u, b\}\} \mapsto \{g, \mu(\{g^{-1}u, b\})\}, \]
that will be called the value of \(\mu\) on \(N\). This definition is correct. Indeed, if we replace \(g\) by \(gh\), we have
\[ \{u, b\} = \{gh, \{h^{-1}g^{-1}u, b\}\} \mapsto \{gh, \mu(\{h^{-1}g^{-1}u, b\})\} = \{g, \mu(\{g^{-1}u, b\})\}. \]

There also exists a pullback version of this concept, in which we replace \(P^r_nS\) by \(P^r_nS_{\ast}\). In the case \(B = pt\) is a singleton, so that \(r = 0\), we obtain the concept of \((n, q)\)-object on \(S\) introduced in Section 4. Indeed, \(P^q_nS[pt] = K^q_nS\).

In particular, a section \(\varphi: K^r_nS \to I^{r:a,b}_{n:c,d}S\) can be interpreted as a linear morphism
\[ \varphi: \bigotimes H^r_nS \otimes \bigotimes H^r_nS \to \bigotimes V^r_nS \otimes \bigotimes V^r_nS. \]

If we consider the induced action of \(G\) on \(I^{r:a,b}_{n:c,d}S\) and set
\[ B = (\bigotimes H^r_nS \otimes \bigotimes H^r_nS)_c \quad \text{and} \quad A = (\bigotimes V^r_nS \otimes \bigotimes V^r_nS)_c, \]
we obtain the concept of invariant section \(\varphi\).

A simple example to Definition 14 is the classical connection on a submanifold \(N\), i.e. a principal connection on \(P^1_N\). This is a second order geometric object field on \(N\), [16]. The problem of finding an invariant construction of induced classical connection on a submanifold is important for both affine and projective differential geometries.
Absolute differentiation in Cartan spaces

References


