Gauge invariance, charge conservation, and variational principles

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Abstract

We present new results on the correspondence between symmetries, conservation laws and variational principles for field equations in general non-abelian gauge theories. Our main result states that second order field equations possessing translational and gauge symmetries and the corresponding conservation laws are always derivable from a variational principle. We also show by the way of examples that the above result fails in general for third order field equations.

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1 Introduction and main results

The classical Noether's theorem establishes essentially a one-to-one correspondence between the symmetries and conservation laws of a system of partial differential equations admitting a variational principle. In 1977, F. Takens [28] considered and rigorously formulated the following novel aspect of Noether's theorem: Let $\mathfrak{g}$ be a Lie algebra of vector fields defined on the space of independent and dependent variables, and suppose that a system of differential equations is invariant under $\mathfrak{g}$ and that each element in $\mathfrak{g}$ generates a conservation law for the system. Does it then follow that the system arises from a variational principle, i.e., that it is the Euler-Lagrange expression of some Lagrangian function? In his original paper Takens considered the question for second order scalar equations, systems of linear equations, and metric field theories. Subsequently, Takens' results on second order scalar equations and on systems of linear equations were substantially generalized by Anderson and Pohjanpelto [3], [4], [26]. We refer to [3] in particular for more background material and motivation on Takens' problem.

Apart from the papers listed above, the literature dealing with the existence of variational principles for systems of differential equations admitting a Lie algebra of symmetries and the corresponding conservation laws is mainly limited to classical field theories, where the symmetry group is the infinite dimensional group of coordinate transformations of the underlying manifold and the conservation laws express the vanishing of the covariant divergence (or some variant of it) of the field equations. The classification results of Cartan [7], Vermeil [29], and Weyl [30] imply that second order quasi-linear field equations for the metric tensor possessing the symmetries and conservation laws of the Einstein equations necessarily arise from a variational principle. This result was later generalized to general second order equations for the metric tensor and to third order equations in the 3-dimensional case by Lovelock, [19, 21], whose results were subsequently extended to metric-scalar [13], metric-vector [20], and metric-bivector [22] theories. More recently, pure vector field theories with the symmetry group consisting of spatial translations and $U(1)$ gauge transformations were treated in [5].

In this paper we investigate the relationship between symmetries, conservation laws, and variational principles for gauge theories with a general structure group on the n-dimensional Euclidean space $\mathbb{R}^n$. Gauge theories with a non-Abelian structure group play a central role in quantum field theories by providing a unified framework for the description of electromagnetism and the weak and strong forces [10], and in geometry as the pivotal ingredient in the Donaldson and Seiberg-Witten theories [8, 23]. Recently, higher order gauge theories, i.e., theories with field equations or order $k \geq 3$, have been developed in classical [9], [17] as well as in quantum settings [11], [12, p. 217].

The primary goal of the present paper is to identify conditions under which a system of gauge field equations admitting translational and gauge symmetries and the associated conservation laws arises from a variational principle. In contrast to the results of Cartan, Vermeil, Weyl, and Lovelock, field equations of gauge theories, even in low orders, are not specified by these conditions, which rules out solving Takens' question in this situation by a brute force classification process. However, as is well known, the vanishing of the classical Helmholtz conditions for a system of differential equations guarantees the existence of a Lagrangian for the system, and our problem is rendered tractable by an analysis of these conditions for systems with the prescribed symmetries and conservation laws.

Write (x^i) for the coordinates on $\mathbb{R}^n$ and let $\{e_\alpha\}$ be a basis of the Lie algebra $\mathfrak{g}$ of an r-dimensional Lie group G with structure constants $c^\alpha_{\beta\gamma}$. A gauge field

$$A = A^\alpha_a(x^i)dx^a \otimes e_\alpha$$

is a $\mathfrak{g}$-valued 1-form on $\mathbb{R}^n$, where the A^α_a stand for the components of A. The gauge field A is subject to a system of k-th order partial differential equations

$$T^a_\alpha = T^a_\alpha(x^i, A^\beta_b, A^\beta_{b,j_1}, A^\beta_{b,j_1j_2}, \ldots, A^\beta_{b,j_1j_2\cdots j_k}) = 0, \qquad a = 1, \ldots, n, \quad \alpha = 1, \ldots, r,$$

where $A^\beta_{b,j_1j_2\cdots j_l}$ denotes the derivative of A^β_b with respect to the independent variables x^{j_1},x^{j_2},..., x^{j_l}. The operator T^a_α is locally variational if it can be written as the Euler–Lagrange expression

$$T^a_\alpha = \mathrm{E}^a_\alpha(L) = \frac{\partial L}{\partial A^\alpha_a} - D_{i_1}\frac{\partial L}{\partial A^\alpha_{a,i_1}} + D_{i_1}D_{i_2}\frac{\partial L}{\partial A^\alpha_{a,i_1i_2}} - \cdots$$

of some locally defined Lagrangian $L = L(x^j, A^\beta_b, A^\beta_{b,j_1}, A^\beta_{b,j_1j_2}, \ldots, A^\beta_{b,j_1j_2\cdots j_l})$, where D_i denotes the standard coordinate total derivative operator.

In this paper we consider the following classes of symmetries and conservation laws for a differential operator T^a_α.

Symmetries

[S1] The operator T^a_α is invariant under the infinitesimal group of spatial translations

$$\mathfrak{t}(n) = \left\{ \mathbf{t} = a^i\frac{\partial}{\partial x^i} \mid (a^i) \in \mathbb{R}^n \right\}. \tag{1}$$

[S2] The operator T^a_α is invariant under the infinite dimensional group of infinitesimal gauge transformations

$$\mathfrak{ga}(n) = \left\{ Q_\varphi = \left(\varphi^\alpha_{,a} + c^\alpha_{\beta\gamma}A^\beta_a\varphi^\gamma\right)\frac{\partial}{\partial A^\alpha_a} \mid \varphi \in C^\infty(\mathbb{R}^n, \mathfrak{g}) \right\}. \tag{2}$$

Conservation laws

[C1] There are functions $t^i_p = t^i_p(x^j, A^\beta_b, A^\beta_{b,j_1}, A^\beta_{b,j_1j_2}, \ldots, A^\beta_{b,j_1j_2\cdots j_l})$ such that, for each $p = 1, 2, \ldots, n$,

$$A^\alpha_{a,p}T^a_\alpha = D_i(t^i_p).$$

[C2] The covariant divergence of the operator T^a_α vanishes identically,

$$\nabla_a T^a_\alpha = D_a T^a_\alpha + c^\gamma_{\alpha\beta}A^\beta_a T^a_\gamma = 0.$$

Our main result is the following.

Theorem 1. *Suppose that the differential operator* T^a_α *has symmetries* [S1], [S2] *and conservation laws* [C1]. *Then* T^a_α *is locally variational if*

1. T^a_α *is of second order,* *or*
2. T^a_α *is polynomial in the components of the gauge field and their derivatives of degree at most* n.

This paper is organized as follows. After covering some preliminary material relevant to the problem at hand in section 2, in section 3 we analyze the relationship between symmetries [S1], [S2] and conservation laws [C1], [C2] for gauge field equations. In particular, we show that any differential operator T^a_α possessing symmetries [S1], [S2] and conservation laws [C1] also necessarily admits conservation laws [C2]. This interesting though elementary fact does not seem to have been previously noted in the literature. In section 4 we present the proof of Theorem 1, and, finally, in section 5, for $n \geq 3$, we employ a general construction to derive examples of third-order differential operators with symmetries [S1], [S2] and conservation laws [C1], [C2] that fail to be locally variational, showing that Theorem 1 is sharp as far as the order of the differential operator T^a_α is concerned.

2 Preliminaries

In this section we collect together some basic definitions and results from the formal calculus of variations on jet spaces most relevant to the problem at hand. For more details and proofs we refer, e.g., to [1, 25].

Let G be an r-dimensional Lie group with Lie algebra $\mathfrak{g}$. We write $\mathcal{A} \to \mathbb{R}^n$ for the bundle of gauge fields with structure group G over the n-dimensional Euclidean space $\mathbb{R}^n$. Fix coordinates (x^i) on $\mathbb{R}^n$ and a basis $\{e_\alpha\}$ for $\mathfrak{g}$, and let A^α_a, $a = 1, \ldots, n$, $\alpha = 1, \ldots, r$, denote the components of the gauge field. Then, as a coordinate bundle, $\mathcal{A} = \{(x^i, A^\alpha_a)\} \to \{(x^i)\}$. We denote the bundle of order k, $0 \leq k \leq \infty$, jets of local sections of $\mathcal{A}$ by $J^k(\mathcal{A})$; in coordinates

$$J^k(\mathcal{A}) = \{(x^i, A^\alpha_a, A^\alpha_{a,j_1}, A^\alpha_{a,j_1 j_2}, \ldots, A^\alpha_{a,j_1 j_2 \cdots j_l}, \ldots, A^\alpha_{a,j_1 j_2 \cdots j_k})\}, \tag{3}$$

where $A^\alpha_{a,j_1 j_2 \cdots j_l}$ stands for the lth order derivative variables. We also use the notation $A^{[l]}$ to collectively denote all variables $A^\alpha_{a,j_1 j_2 \cdots j_p}$, $p = 0, \ldots, l$, up to order l.

Let $I = (i_1, i_2, \ldots, i_k)$, $1 \leq i_l \leq n$, denote an unordered multi-index of length $|I| = k$. Define partial derivative operators $\partial^{a,I}_\alpha$ by

$$\partial^{a,I}_\alpha A^\beta_{b,J} = \begin{cases} \delta^\beta_\alpha \delta^a_b \delta^{(i_1}_{j_1} \delta^{i_2}_{j_2} \cdots \delta^{i_k)}_{j_k}, & \text{if } |I| = |J|, \\ 0, & \text{if } |I| \neq |J|, \end{cases} \tag{4}$$

where round brackets indicate symmetrization in the enclosed indices. Then the standard total derivative operators D_i are given by

$$D_i = \frac{\partial}{\partial x^i} + \sum_{|I|\geq 0} A^{\alpha}_{a,Ii}\partial^{a,I}_{\alpha}, \qquad i = 1, 2, \ldots, n. \tag{5}$$

When there is no danger of confusion we will employ the standard Einstein summation convention in the Euclidean space and Lie algebra indices.

The flow of a vector field

$$X = P^i(x^j, A^{\beta}_b)\frac{\partial}{\partial x^i} + Q^{\alpha}_a(x^j, A^{\beta}_b)\partial^a_{\alpha} \tag{6}$$

on $\mathcal{A}$ induces a transformation on the space of section of $\mathcal{A}$, and, consequently, by differentiation, it induces a local 1-parameter transformation group acting on $J^k(\mathcal{A})$, $k \geq 0$. The associated infinitesimal generator is called the kth-order prolongation of X and is denoted by $\operatorname{pr}^k X$. The components of $\operatorname{pr}^k X$ are given by the usual prolongation formula

$$\operatorname{pr}^k X = P^i D_i + \sum_{|I|\leq k} D_I(X_{\mathrm{ev}}{}^{\alpha}_a)\partial^{a,I}_{\alpha}, \tag{7}$$

where the $X_{\mathrm{ev}}{}^{\alpha}_a$ denote the components of the evolutionary form

$$X_{\mathrm{ev}} = (Q^{\alpha}_a - P^i A^{\alpha}_{a,i})\partial^a_{\alpha}$$

of X and where $D_I = D_{i_1} \cdots D_{i_k}$ for a multi-index $I = (i_1, \ldots, i_k)$. We will also write $\operatorname{pr}^{\infty} X = \operatorname{pr} X$. The vector field (6) is called projectable if the coefficients $P^i = P^i(x^j)$ are functions of the independent variables x^j only. In particular, the infinitesimal generators of translations (1) and gauge transformations (2) form the Lie algebra $\mathfrak{t}(n) \times_s \mathfrak{ga}(n)$ of projectable vector fields acting on $\mathbb{R}^n$

Given a differential operator $T^a_{\alpha} = T^a_{\alpha}(x^i, A^{[k]})$, we associate to it the source form

$$T = T^a_{\alpha} dA^{\alpha}_a \wedge \nu, \tag{8}$$

where $\nu = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ is the volume form on $\mathbb{R}^n$. A vector field X on $\mathcal{A}$ generates a conservation law for the source form T if there are differential functions $t^i = t^i(x^j, A^{[l]})$, $i = 1, \ldots, n$, so that

$$X_{\mathrm{ev}}{}^{\alpha}_a T^a_{\alpha} = D_i t^i. \tag{9}$$

The source form T is said to arise from a variational principle if there is a Lagrangian function $L = L(x^i, A^{[l]})$ such that T is the Euler-Lagrange expression of L, i.e.,

$$T^a_{\alpha} = \mathrm{E}^a_{\alpha}(L) = \sum_{|I|\geq 0} (-1)^{|I|} D_I(\partial^{a,I}_{\alpha} L). \tag{10}$$

We will call $\lambda = L(x^i, A^{[l]})\,\nu$ a Lagrangian n-form and $\mathrm{E}(\lambda) = \mathrm{E}^a_{\alpha}(L)\,dA^{\alpha}_a \wedge \nu$ the Euler-Lagrange form associated with λ. As is well known, the Euler-Lagrange operator commutes with the prolonged action of projectable transformations on $\mathcal{A}$; infinitesimally,

$$\mathrm{E}(\mathcal{L}_{\mathrm{pr}\,X}\lambda) = \mathcal{L}_{\mathrm{pr}\,X}\mathrm{E}(\lambda) \tag{11}$$

for every projectable vector field X and Lagrangian form λ, where $\mathcal{L}$ denotes the standard Lie derivative operator.

As in [5], we define the Helmholtz operator H_T acting on evolutionary vector fields $Y = Y_a^\alpha \partial_\alpha^a$ on $\mathcal{A}$ by

$$\mathrm{H}_T(Y) = \mathcal{L}_{\mathrm{pr}\,Y}T - \mathrm{E}(Y \lrcorner\, T). \tag{12}$$

The components $\mathrm{H}_{\alpha\beta}^{ab,I}$ of H_T are determined by

$$\mathrm{H}_T(Y) = \sum_{|I|\geq 0} (D_I Y_b^\beta)\mathrm{H}_{\alpha\beta}^{ab,I} dA_a^\alpha \wedge \nu, \tag{13}$$

and are explicitly given by

$$\mathrm{H}_{\alpha\beta}^{ab,I} = \partial_\beta^{b,I} T_\alpha^a - (-1)^{|I|}\mathrm{E}_\alpha^{a,I}(T_\beta^b), \qquad |I| \geq 0, \tag{14}$$

where $\mathrm{E}_\alpha^{a,I}$ denotes the higher Euler-Lagrange operators acting on a function F defined on some $J^k(\mathcal{A})$ by

$$\mathrm{E}_\alpha^{a,I}(F) = \sum_{|J|\geq 0} (-1)^{|J|}\binom{|I|+|J|}{|I|} D_J(\partial_\alpha^{a,IJ}F), \qquad |I| \geq 0.$$

Note that if $T = T_\alpha^a(x^i, A^{[k]})dA_a^\alpha \wedge \nu$ is of order k, then $\mathrm{H}_{\alpha\beta}^{ab,I} = 0$ for $|I| > k$ and that for $|I| = 0, \ldots, k$, the components $\mathrm{H}_{\alpha\beta}^{ab,I}$ are of order at most $2k - |I|$.

As is well known, a source form $T = \mathrm{E}(L)$ arising from a variational principle satisfies the Helmholtz conditions $\mathrm{H}_T = 0$, or in components, $\mathrm{H}_{\alpha\beta}^{ab,I} = 0$. Conversely, one can show that if the Helmholtz conditions $\mathrm{H}_T = 0$ are satisfied, then, at least locally, the source form T can be written as the Euler-Lagrange expression of some Lagrangian L; see [1]. We will thus call a source form satisfying the Helmholtz conditions locally variational.

Proposition 2. *Suppose that $X = P^i\partial/\partial x^i + Q_a^\alpha \partial_\alpha^a$ is a projectable vector field on $\mathcal{A}$ and that the source form T is invariant under the prolongation* $\mathrm{pr}\,X$ *of X. Then the components $\mathrm{H}_{\alpha\beta}^{ab,I}$ of the Helmholtz operator H_T of T satisfy the invariance conditions*

$$\mathcal{L}_{\mathrm{pr}\,X}\mathrm{H}_{\alpha\beta}^{ab,I} + \sum_{|J|\geq 0} \mathrm{H}_{\alpha\gamma}^{ac,J}(\partial_\beta^{b,I} Q_{c,J}^\gamma) + \mathrm{H}_{\gamma\beta}^{cb,I}\partial_\alpha^a Q_c^\gamma + \frac{\partial P^j}{\partial x^j}\mathrm{H}_{\alpha\beta}^{ab,I} = 0, \tag{15}$$

where $Q_{c,J}^\gamma$ stands for the $A_{c,J}^\gamma$-component of $\mathrm{pr}\,X$.

Proof. We first compute

$$\begin{aligned}\mathcal{L}_{\operatorname{pr} X}\left(\mathrm{H}_T(Y)\right) &= \mathcal{L}_{\operatorname{pr} X}(\mathcal{L}_{\operatorname{pr} Y}T) - \mathcal{L}_{\operatorname{pr} X}\mathrm{E}(Y \,\lrcorner\, T) \\ &= \mathcal{L}_{\operatorname{pr}[X,Y]}T - \mathrm{E}((\mathcal{L}_{\operatorname{pr} X}Y) \,\lrcorner\, T) = \mathrm{H}_T([X,Y]),\end{aligned} \tag{16}$$

where we used the invariance of T and the fact that the Euler-Lagrange operator is equivariant under the action of the prolongation of a projectable vector field X. Next write $Y = Y^\alpha_a \partial^a_\alpha$. Then on account of (13),

$$\begin{aligned}\mathcal{L}_{\operatorname{pr} X}\left(\mathrm{H}_T(Y)\right) = \sum_{|I|\geq 0} \Bigg(&(\mathcal{L}_{\operatorname{pr} X}\mathrm{H}^{ab,I}_{\alpha\beta})D_I Y^\beta_b + \mathrm{H}^{ab,I}_{\alpha\beta}\mathcal{L}_{\operatorname{pr} X}(D_I Y^\beta_b) \\ &+ \mathrm{H}^{cb,I}_{\gamma\beta}(D_I Y^\beta_b)\partial^a_\alpha Q^\gamma_c + \frac{\partial P^j}{\partial x^j}\,\mathrm{H}^{ab,I}_{\alpha\beta} D_I Y^\beta_b \Bigg) dA^\alpha_a \wedge \nu.\end{aligned} \tag{17}$$

But

$$\begin{aligned}\mathcal{L}_{\operatorname{pr} X} D_I Y^\beta_b &= \mathcal{L}_{\operatorname{pr} X}(\operatorname{pr} Y \,\lrcorner\, \theta^\beta_{b,I}) \\ &= \operatorname{pr}[X,Y] \,\lrcorner\, \theta^\beta_{b,I} + \operatorname{pr} Y \,\lrcorner\, (\mathcal{L}_{\operatorname{pr} X}\theta^\beta_{b,I}) \\ &= D_I[X,Y]^\beta_b + \sum_{|J|\geq 0}(D_J Y^\gamma_c)(\partial^{c,J}_\gamma Q^\beta_{b,I}),\end{aligned} \tag{18}$$

where, as usual, $\theta^\beta_{b,I} = dA^\beta_{b,I} - A^\beta_{b,Ii}dx^i$ denotes a basic contact form on $J^\infty(\mathcal{A})$. Now by virtue of (18), equation (17) becomes

$$\begin{aligned}\mathcal{L}_{\operatorname{pr} X}\left(\mathrm{H}_T(Y)\right) = \sum_{|I|\geq 0} \Bigg(&(\mathcal{L}_{\operatorname{pr} X}\mathrm{H}^{ab,I}_{\alpha\beta})D_I Y^\beta_b + \mathrm{H}^{ab,I}_{\alpha\beta} D_I[X,Y]^\beta_b \\ &+ \sum_{|J|\geq 0}\mathrm{H}^{ac,J}_{\alpha\gamma}(D_I Y^\beta_b)\partial^{b,I}_\beta Q^\gamma_{c,J} + \mathrm{H}^{cb,I}_{\gamma\beta}(D_I Y^\beta_b)\partial^a_\alpha Q^\gamma_c \\ &+ \frac{\partial P^j}{\partial x^j}\,\mathrm{H}^{ab,I}_{\alpha\beta} D_I Y^\beta_b \Bigg) dA^\alpha_a \wedge \nu.\end{aligned} \tag{19}$$

Finally, a comparison of (16) with (19) yields equation (15), as required. QED

The following Lie derivative formula, as established in [1, 3], is central in the proof of our main Theorem.

Proposition 3. *Let T be a source form and X a projectable vector field on $\mathcal{A}$. Then*

$$\mathcal{L}_{\operatorname{pr} X}T = \mathrm{E}(X_{ev} \,\lrcorner\, T) + \mathrm{H}_T(X_{ev}). \tag{20}$$

An extension of the Lie derivative formula (20) to non-projectable, generalized vector fields can be found in [1]. Note that for locally variational source forms T, equation (20) reduces to

$$\mathcal{L}_{\operatorname{pr} X}T = \mathrm{E}(X_{\mathrm{ev}} \,\lrcorner\, T).$$

Now

$$\mathcal{L}_{\mathrm{pr}\,X} T = 0,$$

if the source form is invariant under X, or more precisely, if X is a distinguished symmetry of T, while

$$\mathrm{E}(X_{\mathrm{ev}} \lrcorner T) = 0,$$

if X generates a conservation law for T; see [1]. Thus equation (20) provides a version of Noether's theorem for projectable vector fields expressed directly in terms of the system of differential equations without the explicit use of a Lagrangian.

On the other hand, in the situation of Takens' problem, each X belonging to the Lie algebra $\mathfrak{g}$ of vector fields under consideration is assumed to be a distinguished symmetry of the source form T and to generate a conservation law for T, leading to the conditions $H_T(X_{\mathrm{ev}}) = 0$ for all $X \in \mathfrak{g}$ for the Helmholtz operator of T. A basic objective in the analysis of Takens' problem is to identify on mathematical or physical grounds interesting classes $\mathcal{T}$ of source forms (i.e. differential equations) and symmetry algebras $\mathfrak{g}$ of vector fields so that one will be able to classify all $\mathfrak{g}$-invariant Helmholtz operators H_T corresponding to $T \in \mathcal{T}$ satisfying the conditions $H_T(X_{\mathrm{ev}}) = 0$, $X \in \mathfrak{g}$.

3 Symmetries and conservation laws

In this section we analyze the relationship between symmetries [S1], [S2] and conservation laws [C1], [C2] in gauge field theories. As is well known, the commutation formula (11) and Noether's first and second theorems imply that a source form $T = \mathrm{E}(\lambda)$ that is the Euler-Lagrange expression of a Lagrangian form λ with symmetries [S1], [S2], also possesses symmetries [S1], [S2] and, in addition, conservation laws [C1], [C2]. The following result, which is a generalization of those appearing in [14, 15], is a slight but non-vacuous extension of the above conclusions furnished by the classical Noether's theorems.

Proposition 4. *Suppose that the source form T is locally variational. Then T admits symmetries* [S1] *and* [S2] *if and only if it admits conservation laws* [C1] *and* [C2].

Proof. By assumption the Helmholtz operator H_T of T vanishes, and so equation (20) reduces to

$$\mathcal{L}_{\mathrm{pr}\,X} T = \mathrm{E}(X_{\mathrm{ev}} \lrcorner T). \tag{21}$$

Recall that X generates a conservation law for T if and only if $\mathrm{E}(X_{\mathrm{ev}} \lrcorner T) = 0$. Thus the equivalence of [S1] and [C1] immediately follows from equation (21).

Next, with

$$X = Q_\varphi = \left(\varphi^\alpha_{,a} + c^\alpha_{\beta\gamma} A^\beta_a \varphi^\gamma\right) \frac{\partial}{\partial A^\alpha_a} \in \mathfrak{ga}(n),$$

equation (20) becomes

$$\mathcal{L}_{\mathrm{pr}\,Q_\varphi} T = \mathrm{E}\left(\varphi^\alpha_{,a} T^a_\alpha + c^\alpha_{\beta\gamma} A^\beta_a \varphi^\gamma T^a_\alpha\right).$$

We integrate by parts to write the right-hand side as

$$\mathrm{E}\left(\varphi^\alpha_{,a}T^a_\alpha + c^\alpha_{\beta\gamma}A^\beta_a\varphi^\gamma T^a_\alpha\right) = \mathrm{E}\left(-\varphi^\alpha D_a T^a_\alpha + c^\alpha_{\beta\gamma}A^\beta_a\varphi^\gamma T^a_\alpha\right) = -\mathrm{E}(\varphi^\alpha \nabla_a T^a_\alpha),$$

after which we have

$$\mathcal{L}_{\mathrm{pr}\,Q_\varphi} T = -\mathrm{E}(\varphi^\alpha \nabla_a T^a_\alpha). \tag{22}$$

Equation (22) immediately shows that if T has conservation laws [C2] then it also has symmetries [S2]. It thus remains to prove that the condition

$$\mathrm{E}(\varphi^\alpha \nabla_a T^a_\alpha) = 0 \qquad \text{for all } \varphi \in C^\infty(\mathbb{R}^n, \mathfrak{g}),$$

implies that $\nabla_a T^a_\alpha = 0$.

For this, suppose that for some α_o, $\nabla_a T^a_{\alpha_o} = F(x^i, A^{[l]})$ is of order l and that for some b, β and J with $|J| = l$,

$$(\partial^{b,J}_\beta F)(x^i_o, A^{[l]}_o) \neq 0, \qquad (x^i_o, A^{[l]}_o) \in J^l(\mathcal{A}).$$

Now choose $\varphi_o \in C^\infty(\mathbb{R}^n, \mathfrak{g})$ such that

$$\frac{\partial^{|J|}\varphi^\alpha_o}{\partial x^J}(x^i_o) = \delta^\alpha_{\alpha_o}, \qquad \frac{\partial^{|K|}\varphi^\alpha_o}{\partial x^K}(x^i_o) = 0, \quad K \neq J.$$

Then

$$\begin{aligned}\mathrm{E}^b_\beta\left(\varphi^\alpha_o(\nabla_a T^a_\alpha)\right)(x^i_o, A^{[2l]}_o) &= \sum_{|I|\leq l}(-1)^{|I|}D_I\left(\varphi^\alpha_o \partial^{b,I}_\beta(\nabla_a T^a_\alpha)\right)(x^i_o, A^{[2l]}_o)\\ &= (\partial^{b,J}_\beta F)(x^i_o, A^{[l]}_o) \neq 0,\end{aligned}$$

which is a contradiction. Thus, inductively, we see that $\nabla_a T^a_\alpha = h_\alpha(x^i)$ are functions of x^i only. But due to the translational invariance of the source form T, each h_α must be constant. Finally, by the definition of the covariant divergence (3) and the total derivative operators (5), the covariant divergence $\nabla_a T^a_\alpha$ vanishes when each $A^\alpha_{a,I} = 0$, showing that $h_\alpha = 0$. QED

The next result shows that in gauge field theories, somewhat surprisingly, symmetries [S1], [S2] and conservation laws [C1], [C2] are not mutually independent.

Proposition 5. *Suppose that a source form T possesses symmetries* [S1] *and* [S2] *and conservation laws* [C1]. *Then T also admits conservation laws* [C2].

Proof. First, with $X = Q_\varphi = (\varphi^\alpha_{,a} + c^\alpha_{\beta\gamma}A^\beta_a\varphi^\gamma)\partial/\partial A^\alpha_a \in \mathfrak{ga}(n)$, equation (15) becomes

$$\sum_{|J|\geq 0}\left((D_J Q_{\varphi c}{}^\gamma)\partial^{c,J}_\gamma \mathrm{H}^{ab,I}_{\alpha\beta} + (\partial^{b,I}_\beta D_J Q_{\varphi c}{}^\gamma)\mathrm{H}^{ac,J}_{\alpha\gamma}\right) + (\partial^a_\alpha Q_{\varphi c}{}^\gamma)\mathrm{H}^{cb,I}_{\gamma\beta} = 0. \tag{23}$$

The source form T has symmetries [S1] and conservation laws [C1] and so by equation (20),

$$\sum_{|I|\geq 0} A^{\beta}_{b,Ip}\mathrm{H}^{ab,I}_{\alpha\beta} = 0, \qquad p = 1,\ldots,n. \tag{24}$$

Now apply the vector field $\mathrm{pr}\, Q_\varphi$ to the above equation to see that

$$\sum_{|I|\geq 0}(D_{Ip}Q_{\varphi}{}^{\beta}_{b})\mathrm{H}^{ab,I}_{\alpha\beta} + \sum_{|I|,|J|\geq 0} A^{\beta}_{b,Ip}(D_J Q_{\varphi}{}^{\gamma}_{c})(\partial^{c,J}_{\gamma}\mathrm{H}^{ab,I}_{\alpha\beta}) = 0. \tag{25}$$

Combining (23) and (25) we obtain

$$\sum_{|I|\geq 0}(D_{Ip}Q_{\varphi}{}^{\beta}_{b})\mathrm{H}^{ab,I}_{\alpha\beta} - \sum_{|I|\geq 0} A^{\beta}_{b,Ip}\left(\sum_{|J|\geq 0}(\partial^{b,I}_{\beta}D_J Q_{\varphi}{}^{\gamma}_{c})\mathrm{H}^{ac,J}_{\alpha\gamma} + (\partial^{a}_{\alpha}Q_{\varphi}{}^{\gamma}_{c})\mathrm{H}^{cb,I}_{\gamma\beta}\right) = 0,$$

which, on account of the definition of the total derivative operators (5) and equation (24), simplifies to

$$\sum_{|I|\geq 0}\left(\frac{\partial}{\partial x^p}D_I Q_{\varphi}{}^{\beta}_{b}\right)\mathrm{H}^{ab,I}_{\alpha\beta} = \sum_{|I|\geq 0}\left(D_I Q_{\frac{\partial\varphi}{\partial x^p}}{}^{\beta}_{b}\right)\mathrm{H}^{ab,I}_{\alpha\beta} = 0.$$

But the function $\varphi \in C^\infty(\mathbb{R}^n, \mathfrak{g})$ is arbitrary so we are able to conclude that

$$\sum_{|I|\geq 0}(D_I Q_{\varphi}{}^{\beta}_{b})\mathrm{H}^{ab,I}_{\alpha\beta} = 0 \qquad \text{for all } \varphi \in C^\infty(\mathbb{R}^n, \mathfrak{g}),$$

that is,

$$\mathrm{H}_T(Q_\varphi) = 0 \qquad \text{for all } \varphi \in C^\infty(\mathbb{R}^n, \mathfrak{g}). \tag{26}$$

Due to the gauge invariance of the source form T and equation (26), the Lie derivative formula (20) yields

$$\mathrm{E}\left(\varphi^{\alpha}_{,a}T^{a}_{\alpha} + c^{\alpha}_{\beta\gamma}A^{\beta}_{a}\varphi^{\gamma}T^{a}_{\alpha}\right) = 0.$$

Now we continue as in the second part of the proof of Proposition 4 to conclude that $\nabla_a T^a_\alpha = 0$, as required. QED

4 Proof of Theorem 1

The proof of Theorem 1 relies on the following Lemma, which is a special case of a more general result presented in [2, 24].

Lemma 6. *Let $T = T^a_\alpha dA^\alpha_a \wedge \nu$, $T^a_\alpha = T^a_\alpha(x^i, A^{[k]})$, be a k-th order source form such that the covariant divergence $\nabla_a T^a_\alpha = 0$ vanishes identically. Then the component functions T^a_α are polynomials in the kth order derivative variables $A^\alpha_{a,i_1\cdots i_k}$ of degree at most $n-1$.*

Proof. By assumption,

$$\frac{\partial T^a_\alpha}{\partial x^a} + \sum_{|I|\geq 0} A^\beta_{b,Ia}\partial^{b,I}_\beta T^a_\alpha + c^\gamma_{\alpha\beta}A^\beta_a T^a_\gamma = 0. \tag{27}$$

Now terms in (27) involving the order $k+1$ variables $A^\beta_{b,J}$, $|J| = k+1$, yield the equations

$$\partial^{b,(I}_\beta T^{a)}_\alpha = 0, \qquad |I| = k. \tag{28}$$

Write $\partial^b_{\beta,k,X}$ for the partial differential operator

$$\partial^b_{\beta,k,X} = \sum_{|I|=k} X_I \partial^{b,I}_\beta = \sum_{1\leq i_1,\ldots,i_k\leq n} X_{i_1}\cdots X_{i_k}\partial^{b,i_1\cdots i_k}_\beta, \tag{29}$$

where $X = (X_1,\ldots,X_n) \in (\mathbb{R}^n)^*$ is a covector on $\mathbb{R}^n$. Then equation (28) is equivalent to

$$X_a \partial^b_{\beta,k,X} T^a_\alpha = 0 \qquad \text{for all } X. \tag{30}$$

Next let $G^{b_1}_{\beta_1}\cdots^{b_n}_{\beta_n\,\alpha}$ denote the mappings

$$G^{b_1}_{\beta_1}\cdots^{b_n}_{\beta_n\,\alpha}(X^1,\ldots,X^n,Y) = \partial^{b_1}_{\beta_1,k,X^1}\cdots\partial^{b_n}_{\beta_n,k,X^n}T^a_\alpha Y_a.$$

The operator T^a_α is polynomial in $A^a_{\alpha,I}$, $|I| = k$, of degree at most $n-1$ if and only if the mappings $G^{b_1}_{\beta_1}\cdots^{b_n}_{\beta_n\,\alpha}$ vanish identically. But by (30) and multilinearity, the equation

$$G^{b_1}_{\beta_1}\cdots^{b_n}_{\beta_n\,\alpha}(X^1,\ldots,X^n,Y) = 0$$

holds whenever Y is a linear combination of the covectors X^1, …, X^n. Consequently $G^{b_1}_{\beta_1}\cdots^{b_n}_{\beta_n\,\alpha}$ vanishes for almost all X^1, …, X^n, $Y \in (\mathbb{R}^n)^*$. By continuity, $G^{b_1}_{\beta_1}\cdots^{b_n}_{\beta_n\,\alpha}$ must vanish identically. QED

Proof of Theorem 1. Part 2 of the Theorem is a straightforward consequence of theorem 2.1 in [4]. As to part 1, the source form T also admits conservation law [C2] by Proposition 5. Thus by letting $X = Q_\varphi = (\varphi^\alpha_{,a} + c^\alpha_{\beta\gamma}A^\beta_a\varphi^\gamma)\partial^a_\alpha$ in (20) we see that

$$\begin{aligned}
&\mathrm{H}^{ij,kl}_{\alpha\beta}\varphi^\beta_{,jkl} + (\mathrm{H}^{ij,k}_{\alpha\beta} + c^\gamma_{\zeta\beta}A^\zeta_l\mathrm{H}^{il,jk}_{\alpha\gamma})\varphi^\beta_{,jk}\\
&\quad + (\mathrm{H}^{ij}_{\alpha\beta} + c^\gamma_{\zeta\beta}(A^\zeta_k\mathrm{H}^{ik,j}_{\alpha\gamma} + 2A^\zeta_{k,l}\mathrm{H}^{ik,jl}_{\alpha\gamma}))\varphi^\beta_{,j}\\
&\qquad + c^\gamma_{\zeta\beta}(A^\zeta_j\mathrm{H}^{ij}_{\alpha\gamma} + A^\zeta_{j,k}\mathrm{H}^{ij,k}_{\alpha\gamma} + A^\zeta_{j,kl}\mathrm{H}^{ij,kl}_{\alpha\gamma})\varphi^\beta = 0.
\end{aligned}$$

In particular, we have that

$$\mathrm{H}^{ij}_{\alpha\beta} = c^\gamma_{\beta\zeta}(A^\zeta_k\mathrm{H}^{ik,j}_{\alpha\gamma} + 2A^\zeta_{k,l}\mathrm{H}^{ik,jl}_{\alpha\gamma}), \tag{31}$$

so that it suffices to show that the first and second order Helmholtz conditions of T vanish in order to prove that the source form T is locally variational.

Next, with $X = \partial/\partial x^p$, equation (20) becomes

$$A^{\beta}_{j,p}\mathrm{H}^{ij}_{\alpha\beta} + A^{\beta}_{j,kp}\mathrm{H}^{ij,k}_{\alpha\beta} + A^{\beta}_{j,klp}\mathrm{H}^{ij,kl}_{\alpha\beta} = 0,$$

which, together with (31), yields the equation

$$(A^{\gamma}_{j,kp} + c^{\gamma}_{\beta\zeta}A^{\zeta}_{j}A^{\beta}_{k,p})\mathrm{H}^{ij,k}_{\alpha\gamma} + (A^{\gamma}_{j,klp} + 2c^{\gamma}_{\beta\zeta}A^{\beta}_{k,p}A^{\zeta}_{j,l})\mathrm{H}^{ij,kl}_{\alpha\gamma} = 0. \tag{32}$$

By Lemma 6, the components T^{a}_{α} of T are polynomial in the second order variables $A^{\alpha}_{a,ij}$ of degree $d \le n-1$. Thus, by virtue of the expressions (14), the first order components $\mathrm{H}^{ab,i}_{\alpha\beta}$ are linear in the third order derivative variables with coefficients that are polynomial in $A^{\alpha}_{a,ij}$ of degree at most $d-2$, and the remaining terms in $\mathrm{H}^{ab,i}_{\alpha\beta}$ are polynomial in $A^{\alpha}_{a,ij}$ of degree at most d. Moreover, the second order components $\mathrm{H}^{ab,ij}_{\alpha\beta}$ are of degree at most $d-1$ in the variables $A^{\alpha}_{a,ij}$. So, in particular, when $n = 2$, the first and second order components $\mathrm{H}^{ab,i}_{\alpha\beta}$, $\mathrm{H}^{ab,ij}_{\alpha\beta}$ are functions of $A^{[2]}$, $A^{[1]}$ only, respectively.

Next assume that $n \ge 3$. Apply the operator $\partial^{a}_{\eta,3,X}$ as in (29) to equation (32) to see that

$$(A^{\gamma}_{j,kp} + c^{\gamma}_{\beta\zeta}A^{\zeta}_{j}A^{\beta}_{k,p})\partial^{a}_{\eta,3,X}\mathrm{H}^{ij,k}_{\alpha\gamma} + X_pX_kX_l\mathrm{H}^{ia,kl}_{\alpha\eta} = 0. \tag{33}$$

Denote the degrees of $\partial^{a}_{\alpha,3,X}\mathrm{H}^{ij,k}_{\eta\gamma}$, $\mathrm{H}^{ia,kl}_{\alpha\eta}$ in the variables $A^{\alpha}_{a,ij}$ by d_1, d_2, respectively. It now follows from (33) that $d_1 \le d_2 - 1$. In fact, if $d_1 \ge d_2$, then an application of the differential operator

$$\partial^{b_1}_{\beta_1,2,Y^1}\cdots\partial^{b_{d_1+1}}_{\beta_{d_1+1},2,Y^{d_1+1}}$$

to (33) yields the equation

$$\sum_{s=1}^{d_1+1} Y^s_p B^s = 0, \qquad p = 1,\ldots,n, \tag{34}$$

where

$$B^s = Y^s_k\partial^{b_1}_{\beta_1,2,Y^1}\cdots\widehat{\partial^{b_s}_{\beta_s,2,Y^s}}\cdots\partial^{b_{d_1+1}}_{\beta_{d_1+1},2,Y^{d_1+1}}\partial^{a}_{\eta,3,X}\mathrm{H}^{ib_s,k}_{\alpha\beta_s}. \tag{35}$$

Since $d_1 \le n-3$, it follows from (34) that the expressions B^s vanish when the covectors $Y^1,\ldots,Y^{d_1+1}$ are linearly independent, and thus by continuity, they vanish identically, which yields a contradiction.

Hence we can assume that $d_1 \le d_2 - 1$, and so, in particular, $d_2 \ge 1$. Now apply the d_2-fold differential operator $\partial^{b_1}_{\beta_1,2,Y^1}\cdots\partial^{b_{d_2}}_{\beta_{d_2},2,Y^{d_2}}$ to (33) to derive the expression

$$\sum_{s=1}^{d_2} Y^s_p B^s + X_p N = 0, \qquad p = 1,\ldots,n, \tag{36}$$

where

$$B^s = Y^s_k\partial^{b_1}_{\beta_1,2,Y^1}\cdots\widehat{\partial^{b_s}_{\beta_s,2,Y^s}}\cdots\partial^{b_{d_2}}_{\beta_{d_2},2,Y^{d_2}}\partial^{a}_{\eta,3,X}\mathrm{H}^{ib_s,k}_{\alpha\beta_s},$$
$$N = X_kX_l\partial^{b_1}_{\beta_1,2,Y^1}\cdots\partial^{b_{d_2}}_{\beta_{d_2},2,Y^{d_2}}\mathrm{H}^{ia,kl}_{\alpha\eta}.$$

As above, equation (36) implies that $B^s = 0$, $N = 0$, which again is a contradiction, and we deduce that the second order components $\mathrm{H}^{ab,ij}_{\alpha\beta}$ and the derivative terms $\partial^a_{\eta,3,X}\mathrm{H}^{ij,k}_{\alpha\gamma}$ are functions of $A^{[1]}$ only. But then, once more with the help of (33), we are able to conclude that in all the cases $n \geq 2$, the second order Helmholtz conditions $\mathrm{H}^{ab,ij}_{\alpha\beta} = 0$ are satisfied and that the first order components $\mathrm{H}^{ab,i}_{\alpha\beta}$ must be functions of $A^{[2]}$ only.

With this, equation (32) reduces to

$$(A^\gamma_{j,kp} + c^\gamma_{\beta\zeta}A^\zeta_j A^\beta_{k,p})\mathrm{H}^{ij,k}_{\alpha\gamma} = 0,$$

where, by the above, the components $\mathrm{H}^{ij,k}_{\alpha\gamma} = \mathrm{H}^{ij,k}_{\alpha\gamma}(A^{[2]})$ are polynomial of degree $d \leq n-1$ in the second order variables $A^\alpha_{a,ij}$. Now one can easily repeat the above arguments to show that the first order Helmholtz conditions $\mathrm{H}^{ab,i}_{\alpha\beta} = 0$ are also satisfied. Consequently, the source form T is locally variational. QED

5 Examples

We let $\kappa_{\alpha\beta} = c^\gamma_{\alpha\zeta}c^\zeta_{\beta\gamma}$ denote the components of the Killing form κ of the Lie algebra $\mathfrak{g}$ in a given basis $\{e_\alpha\}$ and we will use $\kappa_{\alpha\beta}$ to lower Lie algebra indices in what follows. The components of the curvature, or field strength, of the gauge field A^α_a are given by

$$f^\alpha_{ab} = A^\alpha_{b,a} - A^\alpha_{a,b} + c^\alpha_{\beta\gamma}A^\beta_a A^\gamma_b. \tag{37}$$

In coordinate expressions a semicolon will indicate a covariant derivative so that, for example, $f^\alpha_{ab;i} = D_i f^\alpha_{ab} + c^\alpha_{\beta\gamma}A^\beta_i f^\gamma_{ab}$. Furthermore, we raise and lower the underlying spatial indices using the Minkowski metric $\eta = \mathrm{diag}(-1, 1, \dots, 1)$.

In analogy with the construction presented in [5], section 4, given a differential operator

$$S\colon J^k(\mathcal{A}) \to \Lambda^2(T\mathbb{R}^n) \otimes \mathfrak{g}^* \tag{38}$$

with components $S^{ab}_\alpha = S^{[ab]}_\alpha(A^{[k]})$, we associate to it a source form T of order $k+1$ on $\mathcal{A}$ with the components

$$T^a_\alpha = \nabla_b S^{ab}_\alpha = D_b S^{ab}_\alpha + c^\gamma_{\alpha\beta}A^\beta_b S^{ab}_\gamma. \tag{39}$$

Lemma 7. *Suppose that the differential operator S has symmetries [S1] and that its components S^{ab}_α satisfy the conditions*

$$c^\beta_{\alpha\gamma}f^\gamma_{ab}S^{ab}_\beta = 0, \qquad \alpha = 1, \dots, r, \tag{40}$$

$$f^\alpha_{ab;p}S^{ab}_\alpha = D_i s^i_p, \qquad p = 1, \dots, n, \tag{41}$$

$$\mathcal{L}_{\mathrm{pr}\,Q_\varphi}S^{ab}_\alpha = -c^\beta_{\alpha\gamma}S^{ab}_\beta\varphi^\gamma, \qquad Q_\varphi \in \mathfrak{ga}(n), \tag{42}$$

where the s^i_p are some differential functions. Then the source form T with components T^a_α as in (39) *admits symmetries* [S1], [S2] *and conservation laws* [C1], [C2].

Proof. The source form T clearly admits translational symmetries [S1]. We compute

$$\nabla_a T^a_\alpha = \nabla_a \nabla_b S^{ab}_\alpha = \nabla_{[a}\nabla_{b]} S^{ab}_\alpha = \tfrac{1}{2}\, c^\beta_{\alpha\gamma} f^\gamma_{ab} S^{ab}_\beta,$$

from which we see that T admits conservation laws [C2] exactly when equation (40) holds true.

Next note that

$$f^\alpha_{ab;p} S^{ab}_\alpha = (D_p f^\alpha_{ab} + c^\alpha_{\beta\gamma} A^\beta_p f^\gamma_{ab}) S^{ab}_\alpha,$$

so, on account of (40), equation (41) holds if and only if

$$(D_p f^\alpha_{ab}) S^{ab}_\alpha = D_i s^i_p, \qquad p = 1, \ldots, n.$$

We compute

$$\begin{aligned}(D_p f^\alpha_{ab}) S^{ab}_\alpha &= 2(-D_b D_p A^\alpha_a + c^\alpha_{\beta\gamma} A^\beta_{a,p} A^\gamma_b) S^{ab}_\alpha \\ &= D_b(-2A^\alpha_{a,p} S^{ab}_\alpha) + 2A^\alpha_{a,p}(D_b S^{ab}_\alpha + c^\beta_{\alpha\gamma} A^\gamma_b S^{ab}_\beta) \\ &= D_b(-2A^\alpha_{a,p} S^{ab}_\alpha) + 2A^\alpha_{a,p} T^a_\alpha,\end{aligned}$$

where we first used (37) and the skew-symmetry of S^{ab}_α in the index pair a, b to derive the expression on the right-hand side of the first line of the equation, and then we integrated by parts to obtain the expression on the second line. Thus if equations (40), (41) hold, then

$$A^\alpha_{a,p} T^a_\alpha = D_b(A^\alpha_{a,p} S^{ab}_\alpha) + \frac{1}{2} D_i s^i_p, \tag{43}$$

and hence T admits conservation laws [C1] with $t^i_p = A^\alpha_{a,p} S^{ai}_\alpha + \frac{1}{2} s^i_p$.

Finally, we recall the general fact that given a $\mathfrak{g}$-valued differential function G^α transforming homogeneously under gauge transformations then its covariant derivatives $\nabla_i G^\alpha = D_i G^\alpha + c^\alpha_{\beta\gamma} A^\beta_i G^\gamma$ also transform homogeneously. Thus, on account of (42), the source form T admits symmetries [S2]. QED

Example 8. In this example we derive translationally invariant non-trivial second order solutions to equations (40), (41), (42), when $n \geq 3$. These, via (38), will furnish examples of third order source forms that admit symmetries [S1], [S2] and conservation laws [C1], [C2] but are not locally variational, showing that Theorem 1 is sharp as regards the order of the source form.

First, due to the skew-symmetry of the constants $c_{\alpha\beta\gamma} = \kappa_{\alpha\nu} c^\nu_{\beta\gamma}$ in α, β, γ, one immediately sees that for any functions $q^{ab} = q^{(ab)}(A^{[k]})$ defined on some $J^k(\mathcal{A})$, $k \geq 0$, the products

$$S^{ab}_\alpha = q^{ab} f^{ab}_\alpha \qquad \text{(no summation in } a, b) \tag{44}$$

satisfy equations (40).

Next, for definiteness, assume that the functions s^i_p in (41) are constant coefficient linear expressions in the independent variables x^i so that equation (41) reduces to

$$f^\alpha_{ab;p} S^{ab}_\alpha = \lambda_p, \qquad p = 1, \ldots, n, \tag{45}$$

where λ_p are some constants. Write $m = n(n+1)/2$ for the dimension of the symmetric product $S^2(T^*\mathbb{R}^n)$. Substitution of expressions (44) into (45) yields an $n \times m$ system of linear equations

$$h_{p,ab}\, q^{ab} = \lambda_p, \qquad p = 1, \ldots, n, \tag{46}$$

for the differential functions q^{ab} with gauge invariant coefficients

$$h_{p,ab} = \sum_{\alpha=1}^{r} f_\alpha^{ab} f^\alpha_{ab;p} \qquad \text{(no summation in } a, b\text{)},$$

that are symmetric in the index pair a, b. Thus, in particular, the invariance condition (42) is automatically satisfied for S_α^{ab} in (44) when the q^{ab} are constructed as functions in the coefficients $h_{p,ab}$ only.

Next let $\mathcal{E}^{a_1b_1\cdots a_mb_m}$ denote a non-zero rank $2m$ constant tensor on $\mathbb{R}^n$ with the symmetries

$$\begin{aligned} \mathcal{E}^{a_1b_1\cdots b_ia_i\cdots a_mb_m} &= \mathcal{E}^{a_1b_1\cdots a_ib_i\cdots a_mb_m}, \qquad \text{and} \\ \mathcal{E}^{a_1b_1\cdots a_jb_j\cdots a_ib_i\cdots a_mb_m} &= -\mathcal{E}^{a_1b_1\cdots a_ib_i\cdots a_jb_j\cdots a_mb_m}. \end{aligned} \tag{47}$$

One can regard $\mathcal{E}^{a_1b_1\cdots a_mb_m}$ as the permutation symbol on $S^2(T^*\mathbb{R}^n)$.

In the case $n = m = 3$, a solution to (46) is furnished by

$$q^{ab} = V^{-1} V^{p,ab} \lambda_p, \tag{48}$$

where

$$\begin{aligned} V &= \frac{1}{6}\, \epsilon^{p_1p_2p_3} \mathcal{E}^{a_1b_1a_2b_2a_3b_3} h_{p_1,a_1b_1} h_{p_2,a_2b_2} h_{p_3,a_3b_3}, \qquad \text{and} \\ V^{p,ab} &= \frac{1}{2}\, \epsilon^{pp_1p_2} \mathcal{E}^{aba_1b_1a_2b_2} h_{p_1,a_1b_1} h_{p_2,a_2b_2}, \end{aligned}$$

and where $\epsilon^{p_1p_2p_3}$ denotes the usual 3-dimensional permutation symbol.

If $n \geq 4$, we set $\lambda_p = 0$, $p = 1, \ldots, n$, in which case solutions to (46) can be constructed from the $n \times n$ minors of the matrix $(h_{p,ab})$ using the standard formula

$$q^{ab} = \mathcal{E}^{aba_1b_1\cdots a_nb_na_{n+1}b_{n+1}\cdots a_{m-1}b_{m-1}} \tau_{a_{n+1}b_{n+1}\cdots a_{m-1}b_{m-1}} h_{1,a_1b_1} \cdots h_{n,a_nb_n}, \tag{49}$$

where we let the coefficient functions $\tau_{a_{n+1}b_{n+1}\cdots a_{m-1}b_{m-1}} = \tau_{a_{n+1}b_{n+1}\cdots a_{m-1}b_{m-1}}(h_{p,ab})$ depend on the gauge invariant quantities $h_{p,ab}$ and have the same symmetries in their indices as $\mathcal{E}^{a_1b_1\cdots a_mb_m}$.

By the above, with q^{ab} as in (48), (49), the components S_α^{ab} in (44) satisfy conditions (40), (41), (42) of Lemma 7, so it only remains to show that the source form T constructed from S fails to be locally variational. To this end we analyze the third order components

$$\mathrm{H}^{11,222}_{\alpha\beta} = \partial^{1,222}_\beta T^1_\alpha + \partial^{1,222}_\alpha T^1_\beta$$

of the Helmholtz operator H_T of T. Keeping in mind that the order of S^{ab}_α in the coordinates (3) is two, we compute

$$\partial^{1,222}_\beta T^1_\alpha = \partial^{1,222}_\beta \nabla_c S^{1c}_\alpha = \partial^{1,22}_\beta S^{12}_\alpha = (\partial^{1,22}_\beta q^{12}) f^{12}_\alpha,$$

which, with the help of (48), (49), yields

$$\mathrm{H}^{11,222}_{\alpha\beta} = 4V^{-2}V^{2,12}V^{p,12}\lambda_p f^{12}_\alpha f^{12}_\beta, \qquad \text{when } n = 3, \tag{50}$$

and

$$\begin{aligned}\mathrm{H}^{11,222}_{\alpha\beta} &= -2\mathcal{E}^{12a_1b_1\cdots a_nb_na_{n+1}b_{n+1}\cdots a_{m-1}b_{m-1}} \\ &\quad\times\left(\frac{\partial}{\partial h_{2,12}}\tau_{a_{n+1}b_{n+1}\cdots a_{m-1}b_{m-1}}\right) h_{1,a_1b_1}\cdots h_{n,a_nb_n} f^{12}_\alpha f^{12}_\beta, \qquad \text{when } n \geq 4.\end{aligned} \tag{51}$$

If the Killing form κ of $\mathfrak{g}$ is non-trivial, so in particular when $\mathfrak{g}$ is semi-simple, we see from (50), (51) that in general, the Helmholtz operator H_T of T is non-vanishing and that, consequently, part 1 of Theorem 1 does not extend to third order source forms when $n \geq 3$. Additionally, equation (51) with $\tau_{a_{n+1}b_{n+1}\cdots a_{m-1}b_{m-1}}$ chosen to be a suitable polynomial in $h_{p,ab}$ shows that when $n \geq 4$, part 2 of Theorem 1 fails for polynomial source forms of degree $d \geq n+2$ in the variables $A^\alpha_{a,I}$.

Remark 9. If the Killing form κ of $\mathfrak{g}$ vanishes identically, then, in particular, $\mathfrak{g}$ must be solvable, and, when $n \geq 3$, examples presented in [4] can be straightforwardly adapted to provide instances of third order non-variational source forms with symmetries [S1], [S2] and conservation laws [C1], [C2]. We will omit the details in the interest of brevity.

6 Conclusions

In this paper we prove that a system of second order gauge field equations admitting translational and gauge symmetries and the associated conservation laws can be written as the Euler-Lagrange expression of some Lagrangian function. We also show that our result is sharp when the underlying space is at least 3 dimensional by constructing explicit examples of non-variational third-order systems with the required symmetries and conservation laws. However, the optimal form of Theorem 1 for gauge fields in 2 dimensions, which include Yang-Mills fields on Riemann surfaces [6] and the physically important theory of vortices [16], remains an open problem.

A worthwhile generalization of the present work would be to extend the symmetry group to include Lorentz transformations or conformal transformations of the underlying Minkowski space. However, results in [4] intimate that with the extended symmetry group, analysis of the associated Helmholtz conditions, with the source form T now being of third order, will become exceedingly intricate. Additionally, it would be a significant problem to determine whether the Lagrangian for a source form, whose existence is guaranteed by Theorem 1, can be chosen to be invariant under the given symmetry group. This question is tantamount to solving the invariant inverse problem of calculus of variations for Yang-Mills fields with the infinite dimensional pseudo-group generated by translations and gauge transformations. Preliminary results in this direction can be found in [27], see also [18].

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