

# The Hessian and Jacobi Morphisms for Higher Order Calculus of Variations

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## Abstract

We formulate higher order variations of a Lagrangian in the geometric framework of jet prolongations of fibered manifolds. Our formalism applies to Lagrangians which depend on an arbitrary number of independent and dependent variables, together with higher order derivatives. In particular, we show that the second variation on critical sections is equal to the vertical differential of the Euler–Lagrange morphism and (up to total divergencies) to its adjoint morphism. These two objects generalize in an invariant way the Hessian and the Jacobi morphisms of a given Lagrangian to the case of higher order Lagrangians. Some examples of classical Lagrangians are provided to illustrate our method.

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## 1 Introduction

An important aspect of mathematics that can be fit into differential geometry is the calculus of variations. This research started with several formulations of calculus of variations on jet spaces (see, e.g., [14, 21] and Appendix 2). Jet spaces are the natural framework for differential equations in differential geometry. Good sources are [4, 19, 23, 24, 25, 26, 31]. It became later evident that the passage from Lagrangians to Euler–Lagrange equations was nothing but a differential of a certain complex [32, 33, 34, 35, 36]: this led to variational sequences and much more.

So far we found that an interesting aspect of Calculus of Variations [7, 8, 9] was not developed in all details from the point of view of geometric formulations on jet spaces: namely, the second and higher variations of Lagrangians. The Lagrangian characterization of the second variation of a Lagrangian in the framework of jet bundles has been considered in [14] and, more recently, in [7, 8, 9, 10]. We stress that in [7, 8, 14] only first-order Lagrangians were considered, while in [9] also a distinguished class of second-order Lagrangians had been studied. In particular, in [7, 8, 9] it was shown how to recast (up to divergencies) the system formed by the Euler–Lagrange equations together with the Jacobi equations for a given Lagrangian as the Euler–Lagrange equations for a *deformed* Lagrangian.

In this paper we provide a geometrical characterization of the second variation of a Lagrangian of arbitrary order in the general case of  $n$  independent variables and  $m$  unknown functions.

The second variation can be written in an infinite number of ways, by adding arbitrary total divergencies. Some preliminary results concerning the Jacobi morphism were obtained in [11, 27]. In the present paper, we exhibit two distinguished representatives in the class of all such forms. Of course, the choice is dictated by existing or envisaged applications. Our representatives have remarkable intrinsic and coordinate interpretations, they show clearly the connection between Hessian and Jacobi tensors, and are ready for applications to arbitrary order Lagrangians, as we show in the examples.

Notice that in literature only the first order case is usually treated: see [6, 10, 15]. Moreover, our formulation has the advantage to be easily generalizable to higher variations. In [10] a Poincaré–Cartan form is used in the first-order case and for one independent variable, but in higher-order calculus of variations the Poincaré–Cartan form is no longer unique, so that the approach would not lead to a unique formulation of variations of higher order Lagrangians. Last but not least, our approach can be further generalized to higher variations of Euler–Lagrange type morphisms, Helmholtz morphisms and all forms of the variational sequence. This is relevant also in view of the role played by the second variation in many geometric contexts. For example, the second variation of the Yang–Mills functional has an algebraic structure which leads in dimension 4 to important geometric consequences about stable Yang–Mills connections, such as local minima of the functional [2, 5]. As well as, is relevant for the study of the theory of stable and unstable minimal submanifolds of a Riemannian manifold [30].

The paper is organized as follows. In section 2 we recall elements of calculus of variations on jet spaces. We use the language of finite order variational sequences, as

developed by Krupka [22].

In section 3 we introduce the notion of  $i$ -variation of a section as an  $i$ -parameter ‘deformation’. The variations that we consider are of general (nonlinear) form [13]. This allows us to pass to infinitesimal variations, i.e., Lie derivatives, in a natural and straightforward way. Then, we introduce variation of forms as derivatives of their pull-back through the variation of a section with respect to the parameters.

Such a notion of variation is applied in section 4. We concentrate ourselves on the computation of the second variation of a Lagrangian. It turns out that, on critical sections (i.e., on solutions of the Euler–Lagrange equations), the second variation equals the vertical differential of the Euler–Lagrange morphism or (up to total divergencies) its adjoint morphism. We say such morphisms to be, respectively, the *Hessian morphism* and the *Jacobi morphism* of the given Lagrangian. We stress that, for a first order Lagrangian, these two morphisms coincide with the standard Hessian and Jacobi maps. Hence, our method generalizes maps related to the second variation to the case of arbitrary order Lagrangians.

In the last section we consider a few well known and significant examples, showing the role played in these cases by the geometric objects above.

Two short Appendices are included, in which we provide a synthetic version of the well-known jet space formulation of variational problems, both for convenience of the less experienced reader and for a better understanding of our motivation and formalism.

## 2 Variational sequences on jets of fibered manifolds

We recall in this section some basic definitions and results from the theory of jet spaces. Complete treatments of this subject, with different characters, can be found in [4, 19, 23, 24, 25, 26, 31]. Our exposition follows more closely [24, 31].

Our framework is a *fibered manifold*  $\pi: \mathbf{Y} \rightarrow \mathbf{X}$ , with  $\dim \mathbf{X} = n$  and  $\dim \mathbf{Y} = n + m$ . We recall that a fibered manifold is just a surjective submersion  $\pi$ ; in other words, fibers of  $\pi$  need not to be mutually diffeomorphic. A *section* of  $\pi$  is defined to be a map  $s: \mathbf{X} \rightarrow \mathbf{Y}$  such that  $\pi \circ s = \text{id}_{\mathbf{X}}$ . We denote by  $V\mathbf{Y} := \ker T\pi \subset T\mathbf{Y}$  the *vertical subbundle* of the tangent bundle  $T\mathbf{Y}$ .

### 2.1 Jet spaces

For  $r \geq 0$  we are concerned with the  $r$ -jet space  $J_r\mathbf{Y}$ . This space is defined as the set of equivalence classes of sections of  $\pi$  having a contact of order at least  $r$  at a given point. Equivalent sections have the same  $p$ -th order differential at the given point,  $p \leq r$ . We set  $J_0\mathbf{Y} \equiv \mathbf{Y}$ .

There are natural projections  $\pi_s^r: J_r\mathbf{Y} \rightarrow J_s\mathbf{Y}$ ,  $r \geq s$ , sending  $r$ -th equivalent sections into  $s$ -th equivalent sections. Moreover, there are obvious natural projections  $\pi^r: J_r\mathbf{Y} \rightarrow \mathbf{X}$ . The spaces  $J_r\mathbf{Y}$  are endowed with a differentiable structure making  $\pi_s^r$  fiber bundles. Among these, it can be proved that  $\pi_{r-1}^r$  are affine fiberings ( $r \geq 1$ ).

Charts on  $\mathbf{Y}$  adapted to  $\pi$  are denoted by  $(x^\lambda, y^i)$ . Greek indices  $\lambda, \mu, \dots$  run from 1 to  $n$  and they label base coordinates, while Latin indices  $i, j, \dots$  run from 1 to  $m$  and label fiber coordinates, unless otherwise specified. We denote by  $(\partial_\lambda, \partial_i)$  and  $(d^\lambda, d^i)$  the local bases of vector fields and 1-forms on  $\mathbf{Y}$  induced by an adapted chart, respectively.

We denote multi-indices of dimension  $n$  by boldface Greek letters such as  $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ , with  $0 \leq \sigma_\mu, \mu = 1, \dots, n$ ; we set  $|\boldsymbol{\sigma}| := \sigma_1 + \dots + \sigma_n$  and  $\boldsymbol{\sigma}! := \sigma_1! \dots \sigma_n!$ . The charts induced on  $J_r \mathbf{Y}$  are denoted by  $(x^\lambda, y_\sigma^i)$ , with  $0 \leq |\boldsymbol{\sigma}| \leq r$ ; in particular, we set  $y_{\mathbf{0}}^i \equiv y^i$ . The local vector fields and forms of  $J_r \mathbf{Y}$  induced by the above coordinates are denoted by  $(\partial_i^\sigma)$  and  $(d_\sigma^i)$ , respectively.

We recall that a section  $s: \mathbf{X} \rightarrow \mathbf{Y}$  can be prolonged to a section  $j_r s: \mathbf{X} \rightarrow J_r \mathbf{Y}$ . If we set  $y^i \circ s = s^i$ , then we have the coordinate expression

$$(j_r s)_\sigma^i := y_\sigma^i \circ j_r s = \partial_\sigma s^i := \frac{\partial^{|\boldsymbol{\sigma}|} s^i}{\partial x^{\sigma_1} \dots \partial x^{\sigma_n}}.$$

If the order of prolongation needed for  $s$  in formulae is clear from context, then we simply denote the prolongation of  $s$  by  $j_s$ .

The jet spaces carry a natural structure, the *Cartan (or contact) distribution*. It is the vector subbundle of  $TJ_r \mathbf{Y}$  generated by vectors which are tangent to submanifolds of the form  $j_r s(\mathbf{X}) \subset J_r \mathbf{Y}$ . We present here a variant of this structure [24].

We consider the natural complementary fibered morphisms over the affine fibering  $J_{r+1} \mathbf{Y} \rightarrow J_r \mathbf{Y}$  induced by *contact maps* on jet spaces

$$D: J_{r+1} \mathbf{Y} \times_{\mathbf{X}} T\mathbf{X} \rightarrow TJ_r \mathbf{Y}, \quad \omega: J_{r+1} \mathbf{Y} \times_{J_r \mathbf{Y}} TJ_r \mathbf{Y} \rightarrow VJ_r \mathbf{Y},$$

with coordinate expressions, for  $0 \leq |\boldsymbol{\sigma}| \leq r$ , given by

$$D = d^\lambda \otimes D_\lambda = d^\lambda \otimes (\partial_\lambda + y_{\sigma+\lambda}^j \partial_j^\sigma), \quad \omega = \omega_\sigma^j \otimes \partial_j^\sigma = (d_\sigma^j - y_{\sigma+\lambda}^j d^\lambda) \otimes \partial_j^\sigma.$$

Here, the map  $D$  is the inclusion of  $T\mathbf{X}$  into  $TJ_r \mathbf{Y}$  through the differential  $Tj_r s$  of any prolonged section  $j_r s$ , while  $\omega := \text{id}_{TJ_r \mathbf{Y}} - D$ . The vector field  $D_\lambda$  is said to be the *total (or formal) derivative*; the forms  $\omega_\sigma^j$  are said to be *contact (or Cartan) forms*. Contact forms annihilate all vectors generated by  $D_\lambda$ . These are tangent to submanifolds of the form  $j_r s(\mathbf{X}) \subset J_r \mathbf{Y}$ . We will use iterated total derivatives. Namely, if  $f: J_r \mathbf{Y} \rightarrow \mathbb{R}$  is a function, then we set  $D_{\sigma, \lambda} f := D_\lambda D_\sigma f$ .

We have the following natural fibered splitting

$$J_{r+1} \mathbf{Y} \times_{J_r \mathbf{Y}} T^* J_r \mathbf{Y} = \left( J_{r+1} \mathbf{Y} \times_{\mathbf{X}} T^* \mathbf{X} \right) \oplus \mathcal{C}_r^*[\mathbf{Y}], \quad (1)$$

where  $\mathcal{C}_r^*[\mathbf{Y}] := \text{im } \omega^*$  is a subbundle of  $J_{r+1} \mathbf{Y} \times_{J_r \mathbf{Y}} T^* J_r \mathbf{Y}$ , and is naturally isomorphic to  $J_{r+1} \mathbf{Y} \times_{J_r \mathbf{Y}} V^* J_r \mathbf{Y}$  (see [24, 31]).

The above splitting induces splittings in the spaces of forms [37]; here and in the sequel we implicitly use identifications between spaces of forms and spaces of bundle morphisms which are standard in the Calculus of Variations (see, e.g. [18, 19, 21]). Namely, let  $\Lambda_r^p$  be the sheaf of  $p$ -forms on  $J_r \mathbf{Y}$ . We introduce the sheaves of *horizontal*

forms  $\mathcal{H}_{r+1,r}^p$ , i.e., of fibered morphisms over  $\pi_r^{r+1}$  and  $\pi^r$  of the type  $\alpha: J_{r+1}\mathbf{Y} \rightarrow \wedge^p T^* J_r \mathbf{Y}$  and  $\beta: J_r \mathbf{Y} \rightarrow \wedge^p T^* \mathbf{X}$ , respectively. Finally, for  $s \leq r$  we introduce the sheaves of *contact forms*  $\mathcal{C}_{r,s}^p$ , i.e., of fibered morphisms over  $\pi_s^r$  of the type  $\alpha: J_r \mathbf{Y} \rightarrow \wedge^p \text{im } \mathcal{C}_s^*[\mathbf{Y}]$ .

The splitting (1) yields naturally the sheaf splitting

$$\mathcal{H}_{r+1,r}^p = \bigoplus_{t=0}^p \mathcal{C}_{r+1,r}^{p-t} \wedge \mathcal{H}_{r+1}^t. \quad (2)$$

Pull-back yields the inclusion  $\Lambda_r^p \subset \mathcal{H}_{(r+1,r)}^p$ . The effect of (2) on  $\Lambda_r^p$  is the following

$$\Lambda_r^p \subset \bigoplus_{t=0}^p \mathcal{C}_r^{p-t} \wedge \bar{\Lambda}_r^t. \quad (3)$$

Here,  $\bar{\Lambda}_r^t := h(\Lambda_r^p)$  for  $0 < p \leq n$  and  $h$  is defined to be the restriction to  $\Lambda_r^p$  of the projection of the above splitting onto the non-trivial summand with the highest value of  $t$ . Moreover,  $\mathcal{C}_r^{p-t}$  is the space of contact forms with values in  $\mathcal{C}_r^*[\mathbf{Y}]$  (which is a bundle over  $J_{r+1}\mathbf{Y}$ ) and coefficients on  $J_r \mathbf{Y}$ . We define also the map  $v := \text{id} - h$ .

In other words, *if  $\alpha$  is a form on  $J_r \mathbf{Y}$ , then its pull-back  $(\pi_r^{r+1})^*(\alpha)$  can be split into a part containing top degree horizontal forms and a part containing more contact factors* (see e.g. [22, 37]).

In coordinates this is achieved by means of the substitutions

$$d^\lambda \rightarrow d^\lambda, \quad d_\sigma^i \rightarrow \omega_\sigma^i + y_{\sigma+\lambda}^i d^\lambda.$$

which allow to express  $\alpha$  in the basis  $(d^\lambda, \omega_\sigma^i)$  at the cost of raising the order of jet.

The above splitting induces also a decomposition of the exterior differential on  $\mathbf{Y}$ ,  $(\pi_r^{r+1})^* \circ d = d_H + d_V$ , where  $d_H$  and  $d_V$  are called the *horizontal* and *vertical differential*, respectively. The action of  $d_H$  and  $d_V$  on functions and 1-forms on  $J_r \mathbf{Y}$  uniquely characterizes  $d_H$  and  $d_V$  (see, e.g., [31] for more details). In particular, we have the coordinate expressions

$$\begin{aligned} d_H f &= D_\lambda f d^\lambda = (\partial_\lambda f + y_{\sigma+\lambda}^i \partial_i^\sigma f) d^\lambda, & d_V f &= \partial_i^\sigma f \omega_\sigma^i, \\ d_H d^\lambda &= 0, & d_H d_\sigma^i &= -d_{\sigma+\lambda}^i \wedge d^\lambda, & d_H \omega_\sigma^i &= -\omega_{\sigma+\lambda}^i \wedge d^\lambda, \\ d_V d^\lambda &= 0, & d_V d_\sigma^i &= d_{\sigma+\lambda}^i \wedge d^\lambda, & d_V \omega_\sigma^i &= 0. \end{aligned}$$

Any fibered isomorphism  $F: \mathbf{Y} \rightarrow \mathbf{Y}$  over  $\text{id}_\mathbf{X}$  admits a prolongation to a fibered isomorphism  $J_r F: J_r \mathbf{Y} \rightarrow J_r \mathbf{Y}$  such that  $J_r F \circ j_r s = j_r(F \circ s)$ .

A vector field  $\xi$  on  $\mathbf{Y}$  is said to be *vertical* if it has values into  $V\mathbf{Y}$ . A vertical vector field can be conveniently prolonged to a vertical vector field  $j_r \xi: J_r \mathbf{Y} \rightarrow V J_r \mathbf{Y}$ . The vector field  $j_r \xi$  is characterized by the fact that its flow is the natural prolongation of the flow of  $\xi$ . In coordinates, if  $\xi = \xi^i \partial_i$  we have  $j_r \xi = D_\sigma \xi^i \partial_i^\sigma$ ,  $0 \leq |\sigma| \leq r$ . Again, if the order of prolongation needed in formulae for  $\xi$  is clear from context, then we simply denote the prolongation of  $\xi$  by  $j\xi$ .

Let  $\alpha \in \mathcal{C}_r^1 \otimes \bar{\Lambda}_r^n$ . Then we can interpret  $\alpha$  as the differential operator

$$\nabla_\alpha: \mathfrak{X}_0 \rightarrow \bar{\Lambda}_r^n, \quad \xi \mapsto j\xi \lrcorner \alpha,$$

where  $\mathfrak{X}_0$  is the space of vertical vector fields  $\xi: \mathbf{Y} \rightarrow V\mathbf{Y}$  and  $\lrcorner$  denotes the inner product. This is an operator in total derivatives and its coordinate expression is  $\nabla_\alpha(\xi^i \partial_i) = D_\sigma \xi^i \alpha_i^\sigma v_{\mathbf{X}}$ , where  $v_{\mathbf{X}} := dx^1 \wedge \dots \wedge dx^n$  is the local volume form induced by a chart  $(x^\lambda)$  on  $\mathbf{X}$ . We can form the *adjoint operator*  $\nabla_\alpha^*: (\bar{\Lambda}_r^n)^* \otimes \bar{\Lambda}_r^n = \mathcal{F}_r \rightarrow \mathfrak{X}_0^* \otimes \bar{\Lambda}_r^n$ . It admits an intrinsic definition (see [4] for details). Its coordinate expression is  $\nabla_\alpha^*(f) = (-1)^{|\sigma|} D_\sigma(\alpha_i^\sigma f) \omega^i \otimes v_{\mathbf{X}}$ . This notion can be extended to a form  $\alpha \in \mathcal{C}_r^1 \otimes \mathcal{C}_0^1 \otimes \bar{\Lambda}_r^n$ : in this case we have  $\nabla_\alpha, \nabla_\alpha^*: \mathfrak{X}_0 \rightarrow \mathcal{C}_0^1 \otimes \bar{\Lambda}_r^n$ , with coordinate expression

$$\nabla_\alpha^*(\xi^i \partial_i) = (-1)^{|\sigma|} D_\sigma(\alpha_{ij}^\sigma \xi^j) \omega^i \otimes v_{\mathbf{X}}. \quad (4)$$

## 2.2 Variational sequence

We recall now shortly the theory of variational sequences on finite order jet spaces, as it was developed by Krupka in [22].

Denote by  $(d \ker h)^s$  the sheaf generated by the presheaf  $d \ker h$ . Set  $\Theta_r^* := \ker h + (d \ker h)^s$ . Then the restriction of exterior differential yields a sheaf sequence  $0 \rightarrow \Theta^*$  which is an exact subsequence of the de Rham sequence of differential forms on  $J_r \mathbf{Y}$ . [22]. Such a subsequence is made by forms that do not produce contribution in action-like functionals [14, 21, 25, 26].

**Definition 2.1** The quotient sequence

$$0 \rightarrow \mathbb{R} \rightarrow \Lambda_r^0 \xrightarrow{\mathcal{E}_0} \Lambda_r^1 / \Theta_r^1 \xrightarrow{\mathcal{E}_1} \Lambda_r^2 / \Theta_r^2 \xrightarrow{\mathcal{E}_2} \dots \xrightarrow{\mathcal{E}_{N-1}} \Lambda_r^N / \Theta_r^N \xrightarrow{\mathcal{E}_N} \Lambda_r^{N+1} \xrightarrow{d} \dots \xrightarrow{d} 0$$

of the de Rham sequence with respect to the contact sequence is called the  $r$ -th order *variational sequence* associated with the fibered manifold  $\mathbf{Y} \rightarrow \mathbf{X}$ . Here the integer  $N$  depends on the dimension of the Cartan distribution on  $J_r \mathbf{Y}$  [22].  $\square$

The variational sequence is locally exact and, due to the ‘abstract’ de Rham Theorem, it has the same cohomology as the de Rham sequence (see [22]).

Pull-back of forms through  $\pi_r^{r+1}$  yields a natural inclusion of the  $r$ -th order variational sequence into the  $(r+1)$ -th order variational sequence. This allows us to represent any equivalence class  $[\alpha] \in \Lambda_r^k / \Theta_r^k$  with a single morphism of bundles over jet spaces, at the cost of raising the order  $r$ . More precisely, the quotient sheaves  $\Lambda_r^k / \Theta_r^k$  in the variational sequence are represented as sheaves of bundle morphisms  $\mathcal{V}_r^k$  (defined on jets of order  $s > r$ ) via the following *intrinsic* isomorphisms [37, 38]

$$I_k: \Lambda_r^k / \Theta_r^k \rightarrow \mathcal{V}_r^k: [\alpha] \mapsto h(\alpha), \quad k \leq n, \quad (5a)$$

$$I_{n+1}: \Lambda_r^{n+1} / \Theta_r^{n+1} \rightarrow \mathcal{V}_r^{n+1}: [\beta] \mapsto E_{h(\beta)}, \quad (5b)$$

$$I_{n+2}: \Lambda_r^{n+2} / \Theta_r^{n+2} \rightarrow \mathcal{V}_r^{n+2}: [\gamma] \mapsto H_{h(\gamma)}. \quad (5c)$$

Let us describe the above morphisms  $h(\alpha)$ ,  $E_{h(\beta)}$ ,  $H_{h(\gamma)}$  and spaces  $\mathcal{V}_r^h$ ,  $h \leq n+2$ .

1.  $h(\alpha)$  is just the horizontalization of  $\alpha$ ; in the case  $k = n$  the morphism  $h(\alpha)$  can be interpreted as a Lagrangian density [1, 14, 18, 19, 21, 23, 26, 33].
2.  $E_{h(\beta)}$  is the Euler–Lagrange morphism associated to  $h(\beta)$ . More precisely, it can be proved [18, 21, 37] that any form  $h(\beta)$  can be uniquely split into the sum

$$(\pi_{r+1}^{2r+1})^* h(\beta) = E_{h(\beta)} - d_H(p_{h(\beta)}) \quad (6)$$

where  $E_{h(\beta)} \in \mathcal{C}_{(2r,0)}^1 \wedge \mathcal{V}_{2r}^n$  and  $p_{h(\beta)} \in \mathcal{C}_{(2r-1,r-1)}^1 \wedge \mathcal{V}_{2r-1}^{n-1}$ . Here  $E_{h(\beta)}$  and  $d_H(p_{h(\beta)})$  are uniquely defined, but  $p_{h(\beta)}$ , the *momentum*, is not; see [18] for a discussion.

3.  $H_{h(\gamma)}$  is the Helmholtz morphism [4, 20, 38] associated to the form  $h(\gamma)$ . More precisely, it can be proved that any form  $h(\gamma)$  can be uniquely split into the sum

$$(\pi_{r+1}^{2r+1})^* h(\gamma) = H_{h(\gamma)} - d_H(q_{h(\gamma)}), \quad (7)$$

where  $H_{h(\gamma)} \in \mathcal{C}_{(2r+1,r)}^1 \wedge \mathcal{C}_{(2r+1,0)}^1 \wedge \mathcal{V}_{2r}^n$  and  $q_{h(\gamma)} \in \mathcal{C}_{2r-1}^2 \wedge \mathcal{V}_{2r-1}^{n-1}$  (here uniqueness is intended in the same way as above), with the additional condition that  $H_{h(\gamma)}$  is skew-adjoint in the first contact factor.

We will use later on the special case  $[(\pi_{r+1}^{2r+1})^* \gamma] = [dE_{h(\beta)}] \in \mathcal{E}_{n+1}(\Lambda_{2r+1}^{n+1}/\Theta_{2r+1}^{n+1})$ . In this case, if we set  $\eta := E_{h(\beta)}$ , then  $H_{d\eta}$  can be introduced as the skew-symmetrization of the morphism  $\tilde{H}_{d\eta} \in \mathcal{C}_{(4r+1,2r+1)}^1 \otimes \mathcal{C}_{(4r+1,0)}^1 \otimes \mathcal{H}_{4r+1}^n$ , which is characterized by

$$\mathcal{E}(i_{j\Xi} d\eta) = j\Xi \lrcorner \tilde{H}_{d\eta} \quad (8)$$

(see [20, 37] for details). We recall that  $H$  and  $\tilde{H}$  have the same kernel; more precisely,  $\tilde{H}_{d\eta} = 0$  if and only if  $H_{d\eta} = 0$  [20].

Let us recall the coordinate expressions.

1.  $h(\alpha) = A v_{\mathbf{X}}$ , where  $A \in \mathcal{C}^\infty(J_{r+1}\mathbf{Y})$  is a ‘special’ polynomial in the derivatives of order  $r + 1$  (in the sense of [28]; see also [37]).
2. Being locally  $h(\beta) = B_i^\sigma \omega_\sigma^i \wedge v_{\mathbf{X}}$ , we have the standard expression of the Euler–Lagrange morphism (see, e.g., [18, 21, 36])

$$E_{h(\beta)} = (-1)^\sigma D_\sigma B_i^\sigma \omega^i \wedge v_{\mathbf{X}}.$$

3. In the simpler case  $[(\pi_{r+1}^{2r+1})^* \gamma] = [dE_{h(\beta)}] \in \mathcal{E}_{n+1}(\Lambda_{2r+1}^{n+1}/\Theta_{2r+1}^{n+1})$ , that we will use later on, we have locally  $dE_{h(\beta)} = \partial_i^\sigma e_j \omega_\sigma^i \wedge \omega^j \wedge v_{\mathbf{X}}$ , where  $e_j = (-1)^\sigma D_\sigma B_j^\sigma$ , so that

$$\begin{aligned} \tilde{H}_{dE_{h(\beta)}} &= H_{ij}^\sigma \omega_\sigma^i \otimes \omega^j \otimes v_{\mathbf{X}}, & H_{dE_{h(\beta)}} &= \frac{1}{2} H_{ij}^\sigma \omega_\sigma^i \wedge \omega^j \wedge v_{\mathbf{X}}, \\ H_{ij}^\sigma &:= \partial_i^\sigma e_j - \sum_{|\rho|=0}^{2r+1-|\sigma|} (-1)^{|\langle \sigma, \rho \rangle|} \binom{|\langle \sigma, \rho \rangle|}{|\rho|} D_\rho \partial_j^{(\sigma, \rho)} e_i \end{aligned} \quad (9)$$

where  $(\sigma, \rho)$  denotes the union of the multi-indices  $\sigma$  and  $\rho$  (see [20, 37, 38]; a local version has been also derived in [22, 31]).

Next, we interpret the above spaces  $\mathcal{V}_r^h$ , with  $h \leq n + 2$ .

1.  $\mathcal{V}_r^k := \bar{\Lambda}_r^k$ ,  $k \leq n$ . We recall that  $\bar{\Lambda}_r^k = h(\Lambda_r^k)$ . So,  $\mathcal{V}_r^n$  can be interpreted as the space of Lagrangians of order  $r + 1$  which are polynomials of ‘special type’ [28] in the higher order derivatives;
2.  $\mathcal{V}_r^{n+1}$  is the space of Euler–Lagrange morphisms associated to forms  $h(\beta)$ ; it is a subspace of  $\mathcal{C}_0^1 \wedge \mathcal{H}_{2r+1}^n$  [37].
3.  $\mathcal{V}_r^{n+2}$  is the space of Helmholtz (or Helmholtz–Sonin) morphisms associated to forms  $h(\gamma)$ .

We can read  $\mathcal{E}_k$  through the above isomorphisms  $I_k$ . We obtain the exact sheaf sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \Lambda_r^0 \xrightarrow{\mathcal{E}_0} \mathcal{V}_r^1 \xrightarrow{\mathcal{E}_1} \dots \xrightarrow{\mathcal{E}_{n+1}} \mathcal{V}_r^{n+2} \longrightarrow \dots \quad (10)$$

It turns out [22, 37] that:

1. if  $\mu \in \mathcal{V}_r^k$ , with  $k \leq n-1$ , and  $\mu = h(\alpha) = I_k(\alpha)$ , with  $\alpha \in \Lambda_r^k$ , then  $\mathcal{E}_k(\mu) = h(d\alpha)$ . Notice that  $h(\pi_r^{r+1*} d\alpha) = h((d_V + d_H)(h(\alpha) + v(\alpha))) = D_H(h(\alpha))$ , hence  $\mathcal{E}_k$  is equal to  $d_H$  up to a pull-back;
2. if  $\lambda \in \mathcal{V}_r^n$  then  $\mathcal{E}_n(\lambda) \in \mathcal{V}_r^{n+1}$  coincides with the standard higher order Euler–Lagrange morphism associated with the Lagrangian  $\lambda$ . We will simply write  $\mathcal{E}$  instead of  $\mathcal{E}_n$ ;
3. if  $\eta \in \mathcal{V}_r^{n+1}$  then  $\mathcal{E}_{n+1}(\eta) \in \mathcal{V}_r^{n+2}$  is the Helmholtz morphism corresponding to the Euler–Lagrange morphism  $\eta$ . The exactness of (10) implies that  $\mathcal{E}_{n+1}(\eta) = 0$  if and only if there exists (locally) a Lagrangian  $\lambda \in \mathcal{V}_r^n$  such that  $\mathcal{E}(\lambda) = \eta$ , i.e.,  $\eta$  is *locally variational*.

**Definition 2.2** Let  $k \leq n + 1$ . We say elements  $\mu \in \mathcal{V}_r^k$  to be *variational forms*.

**Remark 2.3** We observe that the spaces  $\Lambda_r^k / \Theta_r^k$  with  $k \geq n + 2$  do not have in the literature (to our knowledge) any interpretation in terms of standard objects of the calculus of variations. In any case, there is a representation  $I_k$  also for such quotient spaces [38]. It comes from the analogue representation for the variational sequence on infinite order jets [4, p. 192]. Furthermore, variational forms of degree  $k > n + 1$  will not play any role in the rest of the paper.  $\square$

**Remark 2.4** Due to  $\mathcal{E}_k \circ \mathcal{E}_{k-1} = 0$ , Lagrangians of the form  $\lambda \in \mathcal{E}_{n-1}(\mathcal{V}_r^{n-1})$  are variationally trivial (i.e., they have identically vanishing Euler–Lagrange equations). Our aim in this paper is to obtain an intrinsic model for the second and higher order variation. This is achieved in the literature in several ways, each of which differs from the others by a total divergence. Our viewpoint is different: we want to provide a model which does not suffer the above arbitrariness.  $\square$

In view of the above Remark, we factorize a part of the variational sequence as follows:

$$\begin{array}{ccc}
 \mathcal{V}_r^n & \xrightarrow{\mathcal{E}} & \mathcal{V}_r^{n+1} \\
 & \searrow \tilde{\pi} & \nearrow \bar{\mathcal{E}} \\
 & \bar{\mathcal{V}}_r^n &
 \end{array} \tag{11}$$

where  $\bar{\mathcal{V}}_r^n := \mathcal{V}_r^n / \mathcal{E}_{n-1}(\mathcal{V}_r^{n-1})$ ,  $\tilde{\pi}$  is the quotient map and  $\bar{\mathcal{E}}$  is the factor map.

It is important to compute infinitesimal symmetries of objects in the variational sequence. To do this, it is natural to look for vector fields  $X$  such that the *standard Lie derivative operator*  $L_X$  passes to the quotient into the variational sequence.

The jet prolongation  $j\Xi$  of vertical vector fields  $\Xi: \mathbf{Y} \rightarrow V\mathbf{Y}$  preserves the contact structure on jets. Hence, it is easy to see that the standard Lie derivative operator with respect to  $j\Xi$  preserves the contact sequence too. This yields the *new operator*  $\mathcal{L}_{j\Xi}$  on the elements of the variational sequence

$$\mathcal{L}_{j\Xi}: \mathcal{V}_r^k \rightarrow \mathcal{V}_r^k: \tau \mapsto I_k([L_{j\Xi}\alpha]), \tag{12}$$

where  $[\alpha] = I_k^{-1}(\tau)$ . The operator  $\mathcal{L}_{j\Xi}$  is said to be the *variational Lie derivative* [12]. Such an operator allows us to recover several well-known formulae from the calculus of variations (see, e.g., [17]) in a unique picture. We have the following expressions [12]:

1. if  $0 \leq p \leq n-1$  and  $\mu \in \mathcal{V}_r^p$ , then

$$\mathcal{L}_{j\xi}\mu = j\xi \lrcorner d_V \mu; \tag{13a}$$

2. if  $p = n$  and  $\lambda \in \mathcal{V}_r^n$ , then

$$\mathcal{L}_{j\xi}\lambda = \xi \lrcorner \mathcal{E}(\lambda) + d_H(j\xi \lrcorner p d_V \lambda); \tag{13b}$$

3. if  $p = n+1$  and  $\eta \in \mathcal{V}_r^{n+1}$ , then

$$\mathcal{L}_{j\xi}\eta = \mathcal{E}(\xi \lrcorner \eta) + j\xi \lrcorner H_{d\eta}. \tag{13c}$$

We remark that the operator  $\mathcal{L}_{j\Xi}: \mathcal{V}_r^n \rightarrow \mathcal{V}_r^n$  factorizes to  $\bar{\mathcal{V}}_r^n$ , since the standard Lie derivative operator with respect to  $j\Xi$  commutes with  $d_H$ . This fact, combined with (13b), produces the further *new quotient operator*  $\bar{\mathcal{L}}_{j\Xi}$ :

$$\bar{\mathcal{L}}_{j\Xi}: \bar{\mathcal{V}}_r^n \rightarrow \bar{\mathcal{V}}_r^n: \lambda \mapsto \bar{\mathcal{L}}_{j\Xi}\lambda := \Xi \lrcorner \mathcal{E}(\lambda). \tag{14}$$

It is clear that this operator can be interpreted as the *infinitesimal variation operator* of a Lagrangian *up to total divergencies*. In the next section we provide a geometric model for such an operator and its iterated applications.

### 3 Variations of forms

We shall here introduce the *variation* of a form as *infinitesimal multiparameter deformation*. This is realized by taking iterated Lie derivatives of the form with respect to vertical vector fields. In this paper we shall consider only vertical variations. In fact, as it is easy to realize (see appendix 1), variations with respect to projectable vector fields do not change the results, since they just add an horizontal differential (a total divergence) which is not relevant for our results in view of Remark 2.4.

Let  $s : \mathbf{X} \rightarrow \mathbf{Y}$  be a section and  $i > 0$ . Let  $\Xi_1, \dots, \Xi_i$  be vertical vector fields on  $\mathbf{Y}$ . Denote by  $\psi_{t_k}^l$ , with  $1 \leq l \leq i$ , the flows generated by  $\Xi_l$ . Then the map

$$\Gamma(t_1, \dots, t_i) = \psi_{t_i}^i \circ \dots \circ \psi_{t_1}^1 \circ s \quad (15)$$

is said to be the *i-th variation* of  $s$  generated by  $(\Xi_1, \dots, \Xi_i)$ .

Let  $\alpha \in \Lambda_r^k$  and let  $\Gamma$  be an  $i$ -th variation of the section  $s$ . Then the map

$$\Delta^i[\Gamma; s](\alpha) := \frac{\partial^i}{\partial t_1 \dots \partial t_i} \Big|_{t_1, \dots, t_i=0} (j_r \Gamma(t_1, \dots, t_i))^* \alpha \quad (16)$$

is said to be the *i-th variation* of the form  $\alpha$  along the section  $s$ .

The following Lemma states the relation between the  $i$ -th variation of a form and its iterated Lie derivative.

**Lemma 3.1** *Let  $\alpha : J_r \mathbf{Y} \rightarrow \wedge^k T^* J_r \mathbf{Y}$ . Let  $\Gamma$  be an  $i$ -th variation of the section  $s$  generated by variation fields  $\Xi_1, \dots, \Xi_i$ . Then we have*

$$\Delta^i[\Gamma; s](\alpha) = (j_r s)^* L_{j_r \Xi_1} \dots L_{j_r \Xi_i} \alpha. \quad (17)$$

PROOF. By the above definitions, we have

$$\begin{aligned} \Delta^i[\Gamma; s](\alpha) &= \frac{\partial^i}{\partial t_1 \dots \partial t_i} \Big|_{t_1, \dots, t_i=0} [(j_r (\psi_{t_i}^i \circ \dots \circ \psi_{t_1}^1 \circ s))^* \alpha] \\ &= (j_r s)^* \frac{\partial^i}{\partial t_1 \dots \partial t_i} \Big|_{t_1, \dots, t_i=0} [(j_r \psi_{t_1}^1)^* \circ \dots \circ (j_r \psi_{t_i}^i)^*] \alpha \\ &= (j_r s)^* L_{j_r \Xi_1} \dots L_{j_r \Xi_i} \alpha, \end{aligned}$$

where  $\psi_{t_k}^k$  are the vertical flows generated by  $\Xi_k$  and we used the definition of prolongation of a vertical vector field (see section 2).

□ QED

From the above considerations it follows that the definition of variation can be given in terms of Lie derivatives with respect to prolongations of vertical vector fields, without any reference to a given section.

**Definition 3.2** Let  $\Xi_1, \dots, \Xi_i$  be vertical vector fields on  $\mathbf{Y}$ . Then the *variation* of a form  $\alpha \in \Lambda_r^k$  is defined to be the operator

$$\Delta^i[\Xi_1, \dots, \Xi_i](\alpha) := L_{j_r \Xi_1} (L_{j_r \Xi_2} (\dots (L_{j_r \Xi_i} \alpha) \dots)).$$

**Remark 3.3** The variation of an  $(n + h)$ -form along a section  $s$  is clearly zero. In fact, the above Lemma shows that the variation is obtained through a pull-back on  $\mathbf{X}$  via  $s$  and any  $(n + h)$ -form on  $\mathbf{X}$  is zero. So, the definition of variation is trivial for  $(n + h)$ -forms.

Nonetheless, we can use the more general Definition 3.2 of variation in all situations. But we loose the classical interpretation of variation as ‘derivative along a parametrized family of sections’.  $\square$

## 4 Variations of Lagrangians

In this section we restrict our attention to variations of elements of the variational sequence, and, in particular, to Lagrangians. Our task is to compute variations ‘up to variationally trivial forms’. In other words, we want to compute the *quotient variations* of Lagrangians in the variational sequence. Indeed, this is straightforwardly permitted by our definitions:  $i$ -th variations are made by Lie derivatives with respect to prolonged vertical vector fields, and they pass to the quotient in the variational sequence [12].

Moreover, we will devote special attention to variations along sections which are *critical* with respect to a given Lagrangian  $\lambda$ , i.e., sections  $s$  such that  $\mathcal{E}(\lambda) \circ js = 0$ .

### 4.1 Quotient variation in the variational sequence

Let  $\alpha \in \Lambda_r^k$ , with  $k \leq n + 1$ , and  $\Xi_1, \dots, \Xi_i$  be vertical vector fields on  $\mathbf{Y}$ . We have

$$\begin{aligned} I_k([\Delta^i[\Xi_1, \dots, \Xi_i]\alpha]) &= I_k([L_{j\Xi_1}(\dots(L_{j\Xi_i}\alpha)\dots)]) \\ &= \mathcal{L}_{j\Xi_1}(\dots(\mathcal{L}_{j\Xi_i}I_k([\alpha]))\dots), \end{aligned}$$

where  $\mathcal{L}$  stands for the variational Lie derivative (see (13a), (13b), (13c)).

**Definition 4.1** The operator

$$\delta^i[\Xi_1, \dots, \Xi_i]I_k([\alpha]) := \mathcal{L}_{j\Xi_1}(\dots(\mathcal{L}_{j\Xi_i}I_k([\alpha]))\dots)$$

is said to be the *quotient variation* of the variational form  $I_k([\alpha]) \in \mathcal{V}_r^k$  with respect to the vertical vector fields  $\Xi_1, \dots, \Xi_i$  on  $\mathbf{Y}$ .

If  $s: \mathbf{X} \rightarrow \mathbf{Y}$  is a (local) section of  $\pi$ , then the form

$$(js)^*(\delta^i[\Xi_1, \dots, \Xi_i]I_k([\alpha]))$$

is said the *quotient variation* of  $I_k([\alpha])$  *along*  $s$ .

**Remark 4.2** The above definition of quotient variation can be applied to all quotient spaces in the variational sequence (see Remark 2.3). But, if we want to calculate variations of forms along sections, only the variations of forms  $[\alpha] \in \mathcal{V}_r^k$  with  $k \leq n$  are non-trivial (see Remark 3.3). In this work we will just devote ourselves to variations of Lagrangians; nonetheless it would be interesting to investigate variations of elements in  $\mathcal{V}_r^k$  with  $k \leq n$ .

## 4.2 Second variation

In this subsection we fix a Lagrangian  $\lambda \in \mathcal{V}_r^n$  and compute its second quotient variation along a critical section  $s$ .

Let  $\Xi: \mathbf{Y} \rightarrow V\mathbf{Y}$  be a vertical vector field. It is natural to introduce an improved quotient variation on the space  $\bar{\mathcal{V}}_r^n$  (see (11)). In fact, the operator  $\bar{\mathcal{L}}_{j\Xi}$  (14) is equal to the operator  $\mathcal{L}_{j\Xi}$  (13b) up to ‘total divergencies’, i.e., up to  $\mathcal{E}_{n-1}$ -exact (variationally trivial) Lagrangians. We recall (see Eq. (10)) that  $\mathcal{E}_{n-1}$  is equal to  $d_H$  up to a pull-back, or up to higher order variationally trivial terms.

**Definition 4.3** The operator

$$\begin{aligned} \bar{\delta}^i[\Xi_1, \dots, \Xi_i][\lambda] &:= \bar{\mathcal{L}}_{j\Xi_1}(\dots(\bar{\mathcal{L}}_{j\Xi_i}\lambda)\dots) \\ &= \Xi_1 \lrcorner \mathcal{E}(\Xi_2 \lrcorner \mathcal{E}(\dots \Xi_{i-1} \lrcorner \mathcal{E}(\Xi_i \lrcorner \mathcal{E}(\lambda))\dots)) \end{aligned}$$

is said to be the *quotient variation* of the Lagrangian  $\lambda$ .

If  $s: \mathbf{X} \rightarrow \mathbf{Y}$  is a (local) section of  $\pi$ , then the form

$$(js)^*(\bar{\delta}^i[\Xi_1, \dots, \Xi_i][\lambda])$$

is called the *quotient variation* of  $[\lambda]$  along  $s$ . □

Of course, the *first quotient variation* of  $\lambda$  is just  $\bar{\delta}^1[\Xi][\lambda] = \Xi \lrcorner \mathcal{E}(\lambda)$ .

Let  $\Xi_1, \Xi_2$  be two vertical vector fields and let us consider the *second quotient variation* of  $\lambda$

$$\bar{\delta}^2[\Xi_1, \Xi_2][\lambda] = \Xi_1 \lrcorner \mathcal{E}(\Xi_2 \lrcorner \mathcal{E}(\lambda)).$$

We state our main result:

**Theorem 4.4** *The second quotient variation of a Lagrangian  $\lambda$  along a critical section  $s$  is equal to either one of the following intrinsic bundle morphisms, which are self-adjoint:*

1. *the differential  $V\mathcal{E}(\lambda)$  of  $\mathcal{E}(\lambda)$  along the fibres of  $\pi^{2r+1}$ , also known as vertical differential:*

$$V\mathcal{E}(\lambda): J_{2r+1}\mathbf{Y} \rightarrow V^*J_{2r+1}\mathbf{Y} \otimes V^*\mathbf{Y} \otimes \wedge^n T^*\mathbf{X};$$

2. *the adjoint  $V\mathcal{E}(\lambda)^*$  of the vertical differential:*

$$V\mathcal{E}(\lambda)^*: J_{2r+1}\mathbf{Y} \rightarrow V^*J_{4r+2}\mathbf{Y} \otimes V^*\mathbf{Y} \otimes \wedge^n T^*\mathbf{X}$$

(see (4)).

PROOF. In coordinates we have  $\Xi_1 = \Xi_1^i \partial_i$ ,  $\Xi_2 = \Xi_2^j \partial_j$  and  $\lambda = L\omega$ . Then

$$\begin{aligned} \bar{\delta}^2[\Xi_1, \Xi_2][\lambda] &= (-1)^{|\sigma|} \Xi_1^j D_\sigma(\partial_j^\sigma(\Xi_2^i \mathcal{E}(L\omega)_i)) \omega \\ &= (-1)^{|\sigma|} \Xi_1^j D_\sigma(\partial_j^\sigma \Xi_2^i \mathcal{E}(L\omega)_i) \omega + (-1)^{|\sigma|} \Xi_1^j D_\sigma(\Xi_2^i \partial_j^\sigma \mathcal{E}(L\omega)_i) \omega. \end{aligned} \quad (18)$$

If  $s$  is a critical section, then the first summand of  $\bar{\delta}^2[\Xi_1, \Xi_2][\lambda]$  in the right-hand side of (18) vanishes identically. The second summand admits an intrinsic interpretation. In fact, we have  $\mathcal{E}^2(\lambda) = 0 = \tilde{H}_{\mathcal{E}(\lambda)}$  due to the property  $\mathcal{E}_{n+1} \circ \mathcal{E}_n = 0$  of the variational sequence and the fact that  $H$  and  $\tilde{H}$  have the same kernel [20]. Hence we have, from the expression (8)

$$0 = j\Xi_1 \lrcorner j\Xi_2 \lrcorner \tilde{H}_{\mathcal{E}(\lambda)} \quad (19)$$

$$= \Xi_1^i D_\sigma \Xi_2^j \partial_j^\sigma \mathcal{E}(\lambda)_i \omega - (-1)^{|\sigma|} \Xi_1^j D_\sigma(\Xi_2^i \partial_j^\sigma \mathcal{E}(L\omega)_i) \omega, \quad (20)$$

so that the second summand of (18) is equal to the first summand of (20), which is the vertical differential  $V\mathcal{E}(\lambda)$  of  $\mathcal{E}(\lambda)$  (also known as *linearization* [4]) contracted with the prolonged fields  $\Xi_1, \Xi_2$ , namely

$$\Xi_1 \lrcorner j\Xi_2 \lrcorner V\mathcal{E}(\lambda) = \Xi_1^i D_\sigma \Xi_2^j \partial_j^\sigma \mathcal{E}(\lambda)_i \omega. \quad (21)$$

A comparison of the coordinate expressions shows that the second summand of (20) is equal to the adjoint of  $V\mathcal{E}(\lambda)$  (see (4)). More precisely,

$$\Xi_1 \lrcorner j\Xi_2 \lrcorner V\mathcal{E}(\lambda) = \Xi_1 \lrcorner j\Xi_2 \lrcorner (V\mathcal{E}(\lambda))^*. \quad \boxed{QED}$$

**Corollary 4.5** *The morphism  $V\mathcal{E}(\lambda)$  is symmetric along any critical section  $s$ , i.e.*

$$j_{2r+1} s^*(\Xi_1 \lrcorner j\Xi_2 \lrcorner V\mathcal{E}(\lambda)) = j_{2r+1} s^*(\Xi_1 \lrcorner j\Xi_2 \lrcorner (V\mathcal{E}(\lambda))^*).$$

**Definition 4.6** We define the fibered morphisms:

1.  $V\mathcal{E}(\lambda)$  to be the *Hessian morphism* associated with the Lagrangian  $\lambda$ ;
2.  $(V\mathcal{E}(\lambda))^*$  to be the *Jacobi morphism* associated with the Lagrangian  $\lambda$ .

It is not difficult to check that the above definition recovers the definitions given by several authors up to ‘total divergencies’ (see e.g.[13, 25, 31]).

Our formulation however has the following advantages:

1. it is manifestly intrinsic (or covariant);
2. it holds for Lagrangians of arbitrary order, while in literature only the first order case is usually treated;

3. it allows an easy generalization to iterated variations of any order;
4. it allows an easy generalization to all spaces in the variational sequence.

**Remark 4.7** We can compare our approach with the one of Crampin [10]. In that paper the Poincaré–Cartan form is used to achieve an intrinsic formula for the second variation of a first-order Lagrangian ( $r = 1$ ) in the case of one independent variable ( $n = 1$ ). However, as is well-known, Poincaré–Cartan forms are no longer unique in the case of many independent variables ( $n > 1$ ) and higher order Lagrangians ( $r > 1$ ), so that our approach seems to be more suitable in the general case.

## 5 Examples

Here we show by simple but relevant examples that our definition of Jacobi morphism coincides with the standard one and it can also be applied to higher order Lagrangians.

**Example 5.1** (Metric Lagrangian). Here we shall derive the classical Jacobi equation for geodesics within our framework. Let  $(Q, g)$  be a Riemannian manifold, with metric tensor  $g = g_{ab}d^a \otimes d^b$ . The Lagrangian for geodesics is  $\lambda = \frac{1}{2}g_{ab}(q)\dot{q}^a\dot{q}^b dt$  and the Euler–Lagrange equation is given by

$$\mathcal{E}(\lambda)_a = -[g_{ab}\ddot{q}^b + \Gamma_{abc}\dot{q}^b\dot{q}^c] = 0,$$

where  $\Gamma_{abc}$  are Christoffel symbols of the first kind.

The Jacobi equation is then obtained by evaluating the local coordinate expression for the Jacobi morphism given by (21) for  $\mathcal{E}(\lambda)_a$ . It is easy to see that the Jacobi morphism for geodesics is in fact given in local coordinates by

$$\begin{aligned} V\mathcal{E}(\lambda)^* &= -[\partial_a g_{bc}\dot{q}^b + \partial_a \Gamma_{bdc}\dot{q}^b\dot{q}^d]\Xi_2^a \Xi_1^c + \\ &\quad + [\partial_a g_{bc}\dot{q}^b - \partial_c g_{ab}\dot{q}^b]\Xi_2^a \frac{d}{dt}\Xi_1^c + \\ &\quad - \partial_b g_{ac}\dot{q}^b \Xi_2^a \frac{d}{dt}\Xi_1^c - g_{ac}\Xi_2^a \frac{d^2}{dt^2}\Xi_1^c. \end{aligned}$$

Taking into account the Euler–Lagrange equation we get finally the Jacobi equation

$$[\partial_a g_{bc}\Gamma_{de}^b - \partial_c \Gamma_{dea}]\dot{q}^d\dot{q}^e \Xi_1^c - 2\Gamma_{cba}\dot{q}^b \frac{d}{dt}\Xi_1^c - g_{ac}\frac{d^2}{dt^2}\Xi_1^c = 0,$$

which can be recasted in the standard form

$$\nabla_{\dot{\gamma}}^2 \Xi_1 + \text{Riem}(\Xi_1, \dot{\gamma}, \dot{\gamma}) = 0,$$

where  $\gamma$  is any geodesic curve,  $\nabla_{\dot{\gamma}}^2$  denotes the second order covariant derivative along the curve  $\gamma$  and  $\text{Riem}(\Xi_1, \dot{\gamma}, \dot{\gamma})$  is the Riemannian curvature tensor. This agrees of course with [8].

**Example 5.2** (Hilbert–Einstein Lagrangian). Let  $\dim \mathbf{X} = 4$  and  $\mathbf{X}$  be orientable. Let  $Lor(\mathbf{X})$  be the bundle of Lorentzian metrics on  $\mathbf{X}$  (provided that it has global sections). Local fibered coordinates on  $J_2(Lor(\mathbf{X}))$  are  $(x^\lambda; g_{\mu\nu}, g_{\mu\nu,\sigma}, g_{\mu\nu,\sigma\rho})$ .

The Hilbert–Einstein Lagrangian is the form  $\lambda_{HE} \in \mathcal{H}_2^4$  defined by  $\lambda_{HE} = L_{HE}\omega$ , where  $L_{HE} = r \sqrt{g}$ . Here  $r : J_2(Lor(\mathbf{X})) \rightarrow \mathbb{R}$  is the function such that, for any Lorentz metric  $g$ , we have  $r \circ j_2g = R$ , being  $R$  the scalar curvature associated with  $g$  and  $g$  the determinant of  $g$ .

A direct computation of the Euler–Lagrange morphism shows that  $E_{d\lambda_{HE}} = G := Ric - \frac{1}{2} Rg \in \mathcal{C}_{(2,0)}^1 \wedge \mathcal{H}_2^4$ ,  $Ric$  being the Ricci tensor of the metric  $g$ .

The Jacobi equations for the Hilbert–Einstein Lagrangian can be then characterized as the kernel of the adjoint of the linearization morphism  $V\mathcal{E}(\lambda_{HE})$ :

$$\begin{aligned} V\mathcal{E}(\lambda_{HE})_{\alpha}^{*\beta} = & \frac{1}{2}[-\nabla_{\lambda}\nabla^{\lambda}\Gamma_{\alpha}^{\beta} + r^{\beta\lambda}\Gamma_{\lambda\alpha} - r_{\alpha\lambda}\Gamma^{\lambda\beta} - 2R_{\rho\alpha\lambda}^{\beta}\Gamma^{\rho\lambda} + \\ & + \delta_{\alpha}^{\beta}r_{\rho\lambda}\Gamma^{\rho\lambda} + (G_{\alpha}^{\beta} + \frac{1}{2}s\delta_{\alpha}^{\beta})\Gamma + \nabla^{\beta}\nabla_{\lambda}\Gamma_{\alpha}^{\lambda} + \nabla_{\alpha}(g^{\beta\gamma}\nabla_{\lambda}\Gamma_{\gamma}^{\lambda}) - \delta_{\alpha}^{\beta}\nabla_{\lambda}(g^{\lambda\gamma}\nabla_{\rho}\Gamma_{\gamma}^{\rho})] = 0, \end{aligned}$$

which coincide with the classical variation of the Einstein tensor (see, e.g., [3]).

We also notice that in the specific case of the Hilbert–Einstein Lagrangian, the Hessian and the Jacobi morphisms coincide. This is in accordance with the fact that  $V\mathcal{E}(\lambda)$  is self-adjoint (see e.g. [29]).

## 6 Conclusions

We provided an intrinsic formalization of higher variations of a Lagrangian. In the case of the second variation, we gave a new interpretation of the Hessian and Jacobi morphism for Lagrangians of arbitrary order.

Some problems remain open at this point and will be investigated in the future:

- It would be interesting to derive a formula for higher order variations, as well as higher order analogues of Hessian and Jacobi morphisms.
- It would also be worth to compute variations of all variational forms (not only Lagrangians).
- There are branches of Quantum Field Theory in which higher order variations play important roles, like e.g. the Batalin–Vilkoviski theory [16]. Such approaches still need a complete mathematical understanding. The above framework could be well-suited for that purpose: in [16] the Batalin–Vilkoviski theory is formalized through jet bundles. Moreover, higher order variations play in any case a role in the path integral approach to quantization.

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Commutative diagrams have been drawn by Paul Taylor's `diagrams` macro package.

## Appendix 1

In this paper we considered only *vertical* variations. Of course, we know that more general variations could be considered: in some problems of field theory (like the computation of conserved currents) it is interesting to consider Lie derivatives of Lagrangians with respect to *projectable vector fields*. A projectable vector field on  $\mathbf{Y}$  is a pair  $(\Xi, \bar{\Xi})$  such that  $\Xi$  is a vector field on  $\mathbf{Y}$ ,  $\bar{\Xi}$  is a vector field on  $\mathbf{X}$  and  $\Xi$  is a bundle morphism over  $\bar{\Xi}$ . In coordinates,  $\Xi = \bar{\Xi}^\lambda \partial_\lambda + \Xi^i \partial_i$ , where  $\bar{\Xi} = \bar{\Xi}^\lambda \partial_\lambda$ .

However, such general variations do not modify the conclusions of our paper in a significant way: their contribution to the variation of a Lagrangian is in fact a total divergence, so that it has obviously to be neglected in our scheme. This fact is, of course, well-known (see, e.g., [15]), but we recall it here for the sake of completeness. The equation (13b) takes the more general form  $\mathcal{L}_{j_r \Xi}(\lambda) = \Xi_V \lrcorner \mathcal{E}(\lambda) + d_H(j_r \Xi_V \lrcorner p_{d_V \lambda} + \bar{\Xi} \lrcorner \lambda)$  (see e.g. [12]), where  $\Xi_V := \omega(\Xi): J_1 \mathbf{Y} \rightarrow V \mathbf{Y}$  is the *vertical part* of  $\Xi$ . In coordinates,  $\Xi_V = (\Xi^i - y_\lambda^i \bar{\Xi}^\lambda) \partial_i$ .

It follows that our results of section 4 hold practically unchanged by just replacing vertical vector fields with vertical parts of projectable vector fields.

## Appendix 2

Here we shortly recall the formulation of variational problems on jet spaces [14, 17, 18, 21, 23, 26, 31, 33] to help the reader to connect the purely differential setting of variational sequences with the classical integral presentation. Nonetheless, we stress the two approaches (differential and integral) are completely independent, even if the latter provided the motivation to the former from an historical viewpoint.

Suppose that an  $r$ -th Lagrangian  $\lambda \in \mathcal{H}_r^n$  is given. Then the *action* of  $\lambda$  on a section  $s: \mathbf{U} \rightarrow \mathbf{Y}$  ( $\mathbf{U}$  is an oriented open subset of  $\mathbf{X}$  with compact closure and regular boundary) is defined to be the real number

$$\int_{\mathbf{U}} (j_r s)^* \lambda.$$

A vertical vector field  $\xi: \mathbf{Y} \rightarrow V \mathbf{Y}$  defined on  $\pi^{-1}(\mathbf{U})$  and vanishing on  $\pi^{-1}(\partial \mathbf{U})$  is said to be a *variation field*. A section  $s: \mathbf{U} \rightarrow \mathbf{Y}$  is said to be *critical* if, for each variation field with flow  $\phi_t$ , we have  $\delta \int_{\mathbf{U}} (J_r \phi_t \circ j_r s)^* \lambda = 0$ , where  $\delta$  is the derivative with respect to the parameter  $t$  and  $J_r \phi_t: J_r \mathbf{Y} \rightarrow J_r \mathbf{Y}$  is the jet prolongation of the flow  $\phi_t$ . It is easy to see that the previous integral expression is equal to  $\int_{\mathbf{U}} (j_r s)^* L_{u_r} \lambda = 0$  for each variation field  $u$ , where  $u_r: J_r \mathbf{Y} \rightarrow V J_r \mathbf{Y}$  is the  $r$ -th jet prolongation of  $u$  (see the first section). Using equation (13b) together with  $L_{u_r} \lambda = i_{u_r} d\lambda$  and Stokes' Theorem,

we find that the above equation is equivalent to  $\int_U (j_{2r}s)^*(i_u E_{d\lambda}) = 0$  for each variation field  $u$ . Finally, by virtue of the fundamental Lemma of Calculus of Variations the above condition is equivalent to  $(j_{2r}s)^* E_{d\lambda} = 0$ , or, that is the same,  $E_{d\lambda} \circ j_{2r}s = 0$ .

Now the reason of the choice of the sheaf  $\Theta_r^k$  (for  $0 \leq k \leq n$ ) as the first non-trivial sheaf of the contact subsequence is clear: for  $k = n$  the sheaf  $\Theta_r^n$  is made by forms which do not contribute to the action. As for the sheaf  $\Theta_r^{n+1}$  it is easily seen that this is precisely the sheaf of forms that give no contribution to the integral  $\int_U (j_{2r}s)^* i_u E_{d\lambda}$  when added to  $E_{d\lambda}$ .

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