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Finite order formulation of Vinogradov's $\mathcal{C}$-spectral sequence

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Abstract

The $\mathcal{C}$-spectral sequence was introduced by Vinogradov in the late Seventies as a fundamental tool for the study of algebro-geometric properties of jet spaces and differential equations. A spectral sequence arise from the contact filtration of the modules of forms on jet spaces of a fibring (or on a differential equation). In order to avoid serious technical difficulties, the order of the jet space is not fixed, i.e., computations are performed on spaces containing forms on jet spaces of any order.

In this paper we show that there exists a formulation of Vinogradov's $\mathcal{C}$-spectral sequence in the case of finite order jet spaces of a fibred manifold. We compute all cohomology groups of the finite order $\mathcal{C}$-spectral sequence. We obtain a finite order variational sequence which is shown to be naturally isomorphic with Krupka's finite order variational sequence.

Key words: Fibred manifold, jet space, variational bicomplex, variational sequence, spectral sequence.
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Introduction

The framework of this paper is that of the algebro-geometric formalism for differential equations. This branch of mathematics started with early works by Ehresmann and Spencer (mainly inspired by Lie), and was carried out by several people. One fundamental aspect is that a natural setting for dealing with global properties of differential equations (such as symmetries, integrability and calculus of variations) is that of jet spaces (see [17, 18, 21] for jets of fibrings and [13, 26, 27] for jets of submanifolds). Indeed, any differential equation can be regarded as a suitable submanifold of a jet space.

The $\mathcal{C}$*-spectral sequence* was introduced by Vinogradov in the late Seventies [24, 25, 26]. It is a spectral sequence (see [6, 19] for a definition) arising from a particular filtration of the De Rham complex on jet spaces (or on differential equations). Namely, there is a natural distribution on any jet space, the *Cartan distribution*, which consists of tangent planes to the prolongation of any section of the jet space. The space of forms annihilating the Cartan distribution, the *contact forms*, is an ideal of the space of all forms, and yields a filtration of this space by means of its powers. Contact forms have deep meanings in several respects. For example, contact forms yield zero contribution to action-like functionals. Indeed, the $\mathcal{C}$-spectral sequence yields the *variational sequence* as a by-product. The morphisms of the variational sequence are the Euler–Lagrange morphism and other relevant maps from the calculus of variations.

The above formulation has been carried out in the case of infinite jets, in order to avoid serious technical difficulties due to the computation of jet order. This paper provides a formulation of Vinogradov's $\mathcal{C}$-spectral sequence in the case of finite order jet spaces of a fibred manifold. A finite order variational sequence then arise from the formulation. Of course, it was evident since the earliest works that such a structure should exist. But the necessary evaluations were not developed in general situations due to severe technical difficulties.

As one could expect, the direct limit of the finite order formulation yields the infinite order formulation by Vinogradov. Moreover, a recent finite order variational sequence by Krupka has been proposed [15]; it is proved that our finite order formulation recover Krupka's formulation, providing an equivalence between the variational sequences of both cases. As a by-product, it is possible to represent any quotient space of the finite order variational sequence by concrete sections of bundles. This problem is easily solved by means of the intrinsic definition of adjoint operator [5, 26].

The equivalence between Vinogradov's and other infinite order formulations [2, 4, 20] either is evident or has already been proved (see [23, 7] for a comparison between Tulczyjew's and Vinogradov's formulation). The advantage of the algebraic techniques used in this paper is in the simplicity of their generalisations. For example, analogous result for jet spaces of submanifolds of a given manifold could be obtained by a straightforward generalisation.

Summarizing, this paper shows the possibility of computing the order of objects involved at any step of the constructions even in the infinite order formalism. It seems that working with infinite order objects implies no loss of information because the order can always be reconstructed. Moreover, working without a definite order is easier than computing it every time. So, it seems that the best strategy would be to compute it only when the problem being investigated strictly requires it. Examples of such problems are provided in [10, 16].

We finish with some mathematical preliminaries.

Preliminaries

In this paper, manifolds and maps between manifolds are C^∞. All morphisms of fibred manifolds (and hence bundles) will be morphisms over the identity of the base manifold, unless otherwise specified. All modules will be modules of sections of some vector bundles (hence projective modules).

Let V be a vector space such that $\dim V = m$. Suppose that $V = W_1 \oplus W_2$, with $p_1 : V \to W_1$ and $p_2 : V \to W_2$ the related projections. Then, we have the splitting

$$\wedge^m V = \bigoplus_{k+h=m} \wedge^k W_1 \wedge \wedge^h W_2 \ , \tag{1}$$

where $\wedge^k W_1 \wedge \wedge^h W_2$ is the subspace of $\wedge^m V$ generated by the wedge products of elements of $\wedge^k W_1$ and $\wedge^h W_2$.

There exists a natural inclusion $\odot_m L(V,V) \subset L(\wedge^m V, \wedge^m V)$. Then, the projections $p_{k,h}$ related to the above splitting turn out to be the maps

$$p_{k,h} = \binom{m}{k} \odot_k p_1 \odot \odot_h p_2 : \wedge^m V \to \wedge^k W_1 \wedge \wedge^h W_2.$$

Let $V' \subset V$ be a vector subspace, and set $W_1' \stackrel{\text{def}}{=} p_1(V')$, $W_2' \stackrel{\text{def}}{=} p_2(V')$. Then we have

$$V' \subset W_1' \oplus W_2', \tag{2}$$

but the inclusion, in general, is not an equality.

1 Jet spaces

In this section we recall some facts on jet spaces.

Our framework is a fibred manifold

$$\pi : E \to M,$$

with $\dim M = n$ and $\dim E = n + m$.

We deal with the tangent bundle $TE \to E$, the tangent prolongation $T\pi : TE \to TM$ and the vertical bundle $VE \stackrel{\text{def}}{=} \ker T\pi \to E$.

Moreover, for $0 \leq r$, we are concerned with the r-th jet space $J^r\pi$; in particular, we set $J^0\pi \equiv E$. We recall the natural fibrings

$$\pi^r_s : J^r\pi \to J^s\pi, \qquad \pi^r : J^r\pi \to M,$$

and the affine bundle $\pi^r_{r-1} : J^r\pi \to J^{r-1}\pi$, which is associated with the vector bundle $\odot^r T^*M \otimes_{J^{r-1}\pi} VE \to J^{r-1}\pi$ for $0 \leq s \leq r$. A detailed account of the theory of jets can be found in [3, 5, 18, 17, 21, 27].

Charts on E adapted to the fibring are denoted by (x^λ, u^i). Greek indices $\lambda, \mu, \ldots$ run from 1 to n and label base coordinates, Latin indices $i, j, \ldots$ run from 1 to m and label fibre coordinates, unless otherwise specified. We denote by $(\partial/\partial x^\lambda, \partial/\partial u^i)$ and (dx^λ, du^i), respectively, the local bases of vector fields and 1-forms on E induced by an adapted chart.

We denote multi-indices by boldface Greek letters such as $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_k)$, with $0 \leq \sigma_1, \ldots, \sigma_k \leq n$. We also set $|\boldsymbol{\sigma}| \stackrel{\text{def}}{=} k$.

The charts induced on $J^r\pi$ are denoted by $(x^\lambda, u^i_{\boldsymbol{\sigma}})$, where $0 \leq |\boldsymbol{\sigma}| \leq r$ and $u^i_0 \stackrel{\text{def}}{=} u^i$. The local vector fields and forms of $J^r\pi$ induced by the fibre coordinates are denoted by $(\partial/\partial u^i_{\boldsymbol{\sigma}})$ and $(du^i_{\boldsymbol{\sigma}})$, $0 \leq |\boldsymbol{\sigma}| \leq r, 1 \leq i \leq m$, respectively.

A (local) section $s\colon M \to E$ can be prolonged to a section $j_r s\colon M \to E$; if we set $u^i \circ s = s^i$, then we have the coordinate expression

$$(j_r s)^i_{\boldsymbol{\sigma}} = \frac{\partial^{|\boldsymbol{\sigma}|}}{\partial x^{\sigma_1} \ldots \partial x^{\sigma_n}} s^i.$$

A fundamental role is played by the *contact maps* on jet spaces (see [18]). Namely, for $0 \leq r$, we consider the natural inclusion over $J^r\pi$

$$D_{r+1} : J^{r+1}\pi \to T^*M \otimes_{J^{r+1}\pi} TJ^r\pi.$$

This inclusion comes from the fact that, if s is a section of π, then we can identify $j_{r+1}s$ with $Tj_r s$. Then, we have the natural morphism

$$\omega_{r+1} : J^{r+1}\pi \to T^*J^r\pi \underset{J^{r+1}\pi}{\otimes} VJ^r\pi,$$

defined by $\omega_{r+1} = \mathrm{id}_{TJ^r\pi} - D_{r+1}$. We have the coordinate expressions

$$D_r = dx^\lambda \otimes D_\lambda = dx^\lambda \otimes \left(\frac{\partial}{\partial x^\lambda} + u^j_{\boldsymbol{\sigma}\lambda} \frac{\partial}{\partial u^j_{\boldsymbol{\sigma}}} \right),$$
$$\omega_r = \omega^j_{\boldsymbol{\sigma}} \otimes \frac{\partial}{\partial u^j_{\boldsymbol{\sigma}}} = (du^j_{\boldsymbol{\sigma}} - u^j_{\boldsymbol{\sigma}\lambda} dx^\lambda) \otimes \frac{\partial}{\partial u^j_{\boldsymbol{\sigma}}},$$

for $0 \leq |\boldsymbol{\sigma}| \leq r$. We stress that

$$D_r \lrcorner\, \omega_r = \omega_r \lrcorner\, D_r = 0 \tag{3}$$

$$(\omega_r)^2 = \omega_r \qquad (D_r)^2 = D_r \tag{4}$$

The (local) vector field $D_{r\lambda}$ is said to be the (λ-th) *total derivative operator*.

We can regard D_{r+1} and ω^*_{r+1} as the injective fibred morphism over $J^r\pi$

$$D_{r+1} : J^{r+1}\pi \underset{J^r\pi}{\times} TM \to J^{r+1}\pi \underset{J^r\pi}{\times} TJ^r\pi,$$
$$\omega^*_{r+1} : J^{r+1}\pi \underset{J^r\pi}{\times} V^*J^r\pi \to J^{r+1}\pi \underset{J^r\pi}{\times} T^*J^r\pi.$$

We have the remarkable vector subbundles

$$C_{r+1,r} \overset{\text{def}}{=} \operatorname{im} D_{r+1} \subset J^{r+1}\pi \underset{J^r\pi}{\times} TJ^r\pi, \tag{5}$$

$$C^*_{r+1,r} \overset{\text{def}}{=} \operatorname{im} \omega^*_{r+1} \subset J^{r+1}\pi \underset{J^r\pi}{\times} T^*J^r\pi \subset T^*J^{r+1}\pi, \tag{6}$$

yielding the splitting [18]

$$J^{r+1}\pi \underset{J^r\pi}{\times} T^*J^r\pi = \left(J^{r+1}\pi \underset{J^r\pi}{\times} T^*M\right) \oplus \operatorname{im} \omega^*_{r+1}. \tag{7}$$

Finally, we have a natural distribution $\mathcal{C}_r$ on $J^r\pi$ generated by the tangent spaces to the prolongation j_rs of any section s, namely the *Cartan distribution* [3, 5, 18, 17, 21, 27]. It is generated by the vector fields $D_{r-1\lambda}$ and $\partial/\partial u^i_{\boldsymbol{\sigma}}$, with $|\boldsymbol{\sigma}| = r$. This distribution has not to be confused with $C_{r,r-1}$, which is a subbundle of a different vector bundle (see (5)), and is generated by D_λ.

REMARK 1. Both bundles $\mathcal{C}_r$ and $C_{r,r-1}$ are part of chains of tangent projections whose inverse limit is the same, i.e., Cartan distribution on infinite order jets, as it is immediate to show.

2 Vector fields and one–forms on jets

Here we introduce distinguished modules over the ring of functions on a jet space of a certain order. Namely, we denote by $\mathcal{F}_M$ the algebra $\mathcal{C}^\infty(M)$, and by $\mathcal{F}_r$ the algebra $\mathcal{C}^\infty(J^r\pi)$.

We denote by $\mathcal{D}_M$ the module of vector fields on M, by $\mathcal{D}_r$ the $\mathcal{F}_r$-module (and $\mathbb{R}$-Lie Algebra) of vector fields on $J^r\pi$ and by $\mathcal{V}_r \subset \mathcal{D}_r$ the module of vertical vector fields.

It would be desirable to complement $\mathcal{V}_r$ in $\mathcal{D}_r$ with a natural direct summand, like $\mathcal{C}_r$. Unfortunately, this holds only in the case of infinite order jet spaces, where Cartan distribution provides the required summand.

We will encompass this problem by observing that another splitting holds. Let us define a *relative vector field* along π^{r+1}_r to be a map $X : J^{r+1}\pi \to TJ^r\pi$ such that $\tau_{\pi^{r+1}_r} \circ X = \pi^{r+1}_r$. We denote by $\mathcal{D}_{r+1,r}$ the $\mathcal{F}_{r+1}$-module of relative vector fields along π^{r+1}_r. In a similar way, we introduce the $\mathcal{F}_{r+1}$-modules $\mathcal{C}\mathcal{D}_{r+1,r}$ and $\mathcal{V}_{r+1,r}$.

PROPOSITION 2. *We have the splitting*

$$\mathcal{D}_{r+1,r} = \mathcal{C}\mathcal{D}_{r+1,r} \oplus \mathcal{V}_{r+1,r}.$$

The projections on the first and second factor are just contraction by $\mathcal{D}_{r+1}$ *and* ω_{r+1}.

We observe that $\mathcal{C}\mathcal{D}_{r+1,r}$ is locally generated over $\mathcal{F}_{r+1}$ by the sections $D_{r+1\lambda}$. Hence, any relative vector field $X \in \mathcal{D}_{r+1,r}$ can be split as

$$X = \ni_X + \mathcal{C}X,$$

with the coordinate expression $\ni_X = (X^i_{\boldsymbol{\sigma}} - u^i_{\boldsymbol{\sigma}\lambda} X^\lambda)\partial/\partial u^i_{\boldsymbol{\sigma}}$, and $\mathcal{C}X = X^\lambda D_\lambda$. Pull-back yields the following inclusion

$$\pi^{r+1*}_r \mathcal{D}_r \subset \mathcal{D}_{r+1,r},$$

so that the above splitting holds for any vector field $X \in \mathcal{D}_r$. In other words, $\ni_X$ can be regarded as the *vertical part* and $\mathcal{C}X$ as the *horizontal part* of X (see [3, 5, 18, 17, 21, 27]).

We consider the dual situation to the vector field case.

Let us set Λ^1_r to be the $\mathcal{F}_r$-module of 1-forms on $J^r\pi$. We introduce the submodule $\mathcal{H}\Lambda^1_r \subset \Lambda^1_r$ of forms with values in T^*M (*horizontal forms*) and the submodule $\mathcal{C}^1\Lambda^1_r$ of forms $\alpha \in \Lambda^1_r$ such that $(j_r s)^*\alpha = 0$ for all sections s of π (*contact forms*). Note that the space of contact forms is just the space of the annihilators of the Cartan distribution.

It would be desirable to complement $\mathcal{H}\Lambda^1_r$ in Λ^1_r with a natural direct summand, like $\mathcal{C}^1\Lambda^1_r$. Unfortunately, this holds only in the case of infinite order jet spaces, where the annihilator of the Cartan distribution provides the required summand.

We will encompass this problem by observing that another splitting holds. Namely, we define $\Lambda^1_{r+1,r}$, $\mathcal{C}^1\Lambda^1_{r+1,r}$ and $\mathcal{H}\Lambda^1_{r+1,r}$ to be the $\mathcal{F}_{r+1}$-modules of 1-forms on $J^{r+1}\pi$ with respective values in $T^*J^r\pi$, $C^*_{r+1,r}$ and T^*M.

PROPOSITION 3. *We have the splitting*

$$\Lambda^1_{r+1,r} = \mathcal{C}^1\Lambda^1_{r+1,r} \oplus \mathcal{H}\Lambda^1_{r+1,r}.$$

The projections on the first and second factor are just contraction by ω_{r+1} *and* D_{r+1}.

If $\alpha \in \Lambda^1_{r+1,r}$ has the coordinate expression $\alpha = \alpha_\lambda dx^\lambda + \alpha^{\boldsymbol{\sigma}}_i du^i_{\boldsymbol{\sigma}}$ $(0 \leq \boldsymbol{\sigma} \leq r)$, then

$$D_{r+1}(\alpha) = (\alpha_\lambda + u^i_{\boldsymbol{\sigma}\lambda}\alpha^{\boldsymbol{\sigma}}_i)\; dx^\lambda, \qquad \omega_{r+1}(\alpha) = \alpha^{\boldsymbol{\sigma}}_i \omega^i_{\boldsymbol{\sigma}}.$$

3 Main splitting

Here, for $k \leq 0$, we consider the standard $\mathcal{F}_r$-module Λ^k_r of k-forms on $J^r\pi$ (which coincides with the exterior power $\wedge^k\Lambda^1_r$).

For $\Lambda^*_r = \oplus_k \Lambda^k_r$ we introduce the ideal $\mathcal{C}^p\Lambda^*_r$ of Λ^*_r generated by p-th exterior powers of $\mathcal{C}^1\Lambda^*_r$ (*p-contact k-forms*). Of course we set $\mathcal{C}^p\Lambda^k_r = \mathcal{C}^p\Lambda^*_r \cap \Lambda^k_r$

We also introduce $\mathcal{H}\Lambda^k_r$ (*horizontal k-forms*) of $\mathcal{H}\Lambda^1_r$. Finally, we consider the obviously defined $\mathcal{F}_{r+1}$-modules $\Lambda^k_{r+1,r}$, $\mathcal{C}^p\Lambda^k_{r+1,r}$ and $\mathcal{H}\Lambda^k_{r+1,r}$.

REMARK 4. We stress that pull-back via π^{r+1}_r yields the inclusion $\Lambda^k_r \subset \Lambda^k_{r+1,r}$, and analogously for $\mathcal{C}^p\Lambda^k_r$ and $\mathcal{H}\Lambda^k_r$.

If $\alpha \in \Lambda^k_{r+1,r}$, then we have

$$\alpha = \alpha^{\boldsymbol{\sigma}_1\dots\boldsymbol{\sigma}_h}_{i_1\dots i_h}{}_{\lambda_{h+1}\dots\lambda_k} du^{i_1}_{\boldsymbol{\sigma}_1} \wedge \dots \wedge du^{i_h}_{\boldsymbol{\sigma}_h} \wedge dx^{\lambda_{h+1}} \wedge \dots \wedge dx^{\lambda_k}.$$

If $\beta \in \mathcal{H}\Lambda^k_{r+1,r}$, then

$$\beta = \beta_{\lambda_1\dots\lambda_k}\, dx^{\lambda_1} \wedge \dots \wedge dx^{\lambda_k}.$$

If $\gamma \in \mathcal{C}^k\Lambda^k_{r+1,r}$, then

$$\gamma = \gamma^{\boldsymbol{\sigma}_1\dots\boldsymbol{\sigma}_k}_{i_1\dots i_k}\, \omega^{i_1}_{\boldsymbol{\sigma}_1} \wedge \dots \wedge \omega^{i_k}_{\boldsymbol{\sigma}_k}, \qquad 0 \leq |\boldsymbol{\sigma}_1|, \dots, |\boldsymbol{\sigma}_k| \leq r.$$

Here, the coordinate functions are sections of Λ^0_{r+1}, and the indices' range is $0 \leq |\boldsymbol{\sigma}_j| \leq r$, $0 \leq h \leq k$. We remark that, in the coordinate expression of α, the indices λ_j are suppressed if $h = k$, and the indices ${}^{\boldsymbol{\sigma}_j}_{i_j}$ are suppressed if $h = 0$.

In the rest of this section, we shall consider the effects of the splitting of proposition 3 on Λ^k_r. As one can expect, Λ^k_r does not split as a direct sum of exterior products of $\mathcal{C}^p\Lambda^k_r$ and $\mathcal{H}\Lambda^q_r$, for suitable p and q. But we have the following result.

PROPOSITION 5. *The splitting of $\Lambda^1_{r+1,r}$ (proposition 3) induces the splitting*

$$\Lambda^k_{r+1,r} = \bigoplus_{l=0}^{k} \mathcal{C}^{k-l}\Lambda^{k-l}_{r+1,r} \wedge \mathcal{H}\Lambda^l_{r+1}$$

We recall that, in the above splitting, direct summands with $l > n$ vanish.

DEFINITION 6. We define the above splitting to be the $\mathcal{C}$-splitting.

We set $p_{k-l,l}$ to be the projection of the $\mathcal{C}$-splitting on the summand $\mathcal{C}^{k-l}\Lambda^{k-l}_{r+1,r} \wedge \mathcal{H}\Lambda^l_{r+1}$. We set also $H = p_{0,k}$ and $V = \mathrm{id} - H$. Due to results in the introduction, we have the following theorem.

PROPOSITION 7. *The explicit expression of the projections of the $\mathcal{C}$-splitting is*

$$p_{k-l,l} = \binom{k}{k-l} \odot_{k-l} \omega_{r+1} \odot \odot_l D_{r+1}.$$

REMARK 8. We have the coordinate expression

$$p_{k-l,l}(\alpha) = \sum u^{j_1}_{\boldsymbol{\tau}_1\lambda_1} \dots u^{j_s}_{\boldsymbol{\tau}_s\lambda_s} \alpha^{\boldsymbol{\sigma}_1 \widehat{\dots} \boldsymbol{\sigma}_{k-l+s} \boldsymbol{\tau}_1 \dots \boldsymbol{\tau}_s}_{i_1 \widehat{\dots} i_{k-l+s}\, j_1 \dots j_s}{}_{\lambda_{s+1}\dots\lambda_l}$$
$$\omega^{i_1}_{\boldsymbol{\sigma}_1} \wedge \widehat{\dots} \wedge \omega^{i_{k-l+s}}_{\boldsymbol{\sigma}_{k-l+s}} \wedge dx^{\lambda_1} \wedge \dots \wedge dx^{\lambda_l},$$

where $0 \leq s \leq l$ and the sum is extended to all subsets $\{{}^{j_1}_{\boldsymbol{\tau}_1} \dots {}^{j_s}_{\boldsymbol{\tau}_s}\}$ of $\{{}^{i_1}_{\boldsymbol{\sigma}_1} \dots {}^{i_{k-l+s}}_{\boldsymbol{\sigma}_{k-l+s}}\}$, where $\widehat{\dots}$ stands for suppressed indexes (and corresponding contact forms) belonging to one of the above subsets.

It turns out that, for $k \leq n$, we have the coordinate expression

$$H(\alpha) = u^{i_1}_{\boldsymbol{\sigma}_1\lambda_1} \dots u^{i_h}_{\boldsymbol{\sigma}_h\lambda_h} \alpha^{\boldsymbol{\sigma}_1\dots\boldsymbol{\sigma}_h}_{i_1\dots i_h}{}_{\lambda_{h+1}\dots\lambda_k} dx^{\lambda_1} \wedge \dots \wedge dx^{\lambda_k},$$

with $0 \leq h \leq k$.

Now, we apply the conclusion of remark 2 of introduction to the $\mathcal{F}_r$-submodule $\Lambda^k_r \subset \mathcal{H}\Lambda^k_{r+1,r}$. To this aim, we want to find the image of Λ^k_r under the projections of the $\mathcal{C}$-splitting. We need to introduce further spaces.

We set $\mathcal{H}_P\Lambda^k_{r+1}$ to be the $\mathcal{F}_r$-module of horizontal forms on $J^{r+1}\pi$ which are polynomials of degree k with respect to the affine structure of $J^{r+1}\pi \to J^r\pi$.

Moreover, we introduce the subspace $\mathcal{C}^k\Lambda^k_{r,r+1} \subset \Lambda^k_{r+1,r}$ of k-forms with values in C^*_{r+1} and coefficients in $\mathcal{F}_r$.

Finally, we denote the restrictions of H, V to Λ^k_r by h, v.

PROPOSITION 9. *Let* $0 < k \leq n$, *and denote*

$$\bar{\Lambda}^k_r \stackrel{def}{=} h(\Lambda^k_r).$$

Then, we have the inclusion $\bar{\Lambda}^k_r \subset \mathcal{H}_P\Lambda^k_{r+1}$.

Moreover, the $\mathcal{F}_r$*-module* $\bar{\Lambda}^k_r$ *admits the following characterization:* $\alpha \in \mathcal{H}_P\Lambda^k_{r+1}$ *belongs to* $\bar{\Lambda}^k_r$ *if and only if there exists* $\beta \in \Lambda^k_r$ *such that* $(j_r s)^*\beta = (j_{r+1}s)^*\alpha$ *for each section* $s : M \to E$.

Proof. If $s : M \to E$ is a section, then the following identities

$$(j_r s)^*\beta = (j_{r+1}s)^* h(\beta), \qquad (j_{r+1}s)^* v(\beta) = 0,$$

yield

$$\alpha = h(\beta) \quad \Leftrightarrow \quad (j_r s)^*\beta = (j_{r+1}s)^*\alpha$$

for all $\alpha \in \mathcal{H}_P\Lambda^k_{r+1}$ and $\beta \in \Lambda^k_r$. QED

REMARK 10. It comes from the above proposition that not any section of $\mathcal{H}_P\Lambda^k_{r+1}$ is a section of $\bar{\Lambda}^k_r$; indeed, a section of $\mathcal{H}_P\Lambda^k_{r+1}$ in general contains 'too many monomials' with respect to a section of $\bar{\Lambda}^k_r$. This can be seen by means of the following example. Consider a one-form $\beta \in \Lambda^1_0$. Then we have the coordinate expressions

$$\beta = \beta_\lambda d^\lambda + \beta_i d^i, \qquad h(\beta) = (\beta_\lambda + u^i_\lambda \beta_i) d^\lambda.$$

If $\alpha \in \mathcal{H}_P\Lambda^1_1$, then we have the coordinate expression $\alpha = (\alpha_\lambda + u^j_\mu \alpha^{\mu}_{j\,\lambda}) d^\lambda$. It is evident that, in general, there does not exist $\beta \in \Lambda^1_0$ such that $h(\beta) = \alpha$.

COROLLARY 11. *Let* $\dim M = 1$. *Then we have*

$$\bar{\Lambda}^1_r = \mathcal{H}_P\Lambda^1_{r+1}.$$

LEMMA 12. *The* $\mathcal{F}_{r+1}$*-module morphisms* H, V *restrict on* Λ^k_r *to the surjective* $\mathcal{F}_r$*-module morphisms*

$$h : \Lambda^1_r \to \bar{\Lambda}^1_r, \qquad v : \Lambda^1_r \to \mathcal{C}^1\Lambda^1_{r,r+1}.$$

Proof. The restriction of H has already been studied. As for the restriction of V, it is easy to see by means of a partition of the unity that it is surjective on $\mathcal{C}^1\Lambda^1_{r,r+1}$. QED

THEOREM 13. *The $\mathcal{C}$-splitting yields the inclusion*

$$\Lambda^k_r \subset \bigoplus_{l=0}^{k} \mathcal{C}^{k-l}\Lambda^{k-l}_{r,r+1} \wedge \bar{\Lambda}^l_r,$$

and the splitting projections restrict to surjective maps.

Proof. In fact, for any $l \leq k$ the restriction of any projection of the $\mathcal{C}$-splitting to Λ^k_r is valued in the above spaces. Let us prove the surjectivity. Let $\Delta \in \mathcal{C}^{k-l}\Lambda^{k-l}_{r,r+1} \wedge \bar{\Lambda}^l_r$, where $0 \leq l \leq n$. We have the coordinate expression

$$\Delta = u^{j_1}_{\boldsymbol{\tau}_1\lambda_1} \dots u^{j_h}_{\boldsymbol{\tau}_h\lambda_h} \Delta^{\boldsymbol{\sigma}_1 \dots \boldsymbol{\sigma}_{k-l}\boldsymbol{\tau}_1 \dots \boldsymbol{\tau}_h}_{i_1 \dots i_{k-l}\, j_1 \dots j_h}{}_{\lambda_{h+1}\dots\lambda_l} \\ \omega^{i_1}_{\boldsymbol{\sigma}_1} \wedge \dots \wedge \omega^{i_{k-l}}_{\boldsymbol{\sigma}_{k-l}} \wedge dx^{\lambda_1} \wedge \dots \wedge dx^{\lambda_l},$$

where $0 \leq |\boldsymbol{\sigma}_i|, |\boldsymbol{\tau}_i| \leq r$ and $0 \leq h \leq l$. If $\{\psi_i\}$ is a partition of the unity subordinate to a coordinate atlas of E, then let

$$\alpha_{\Delta i} \overset{\text{def}}{=} \psi_i\, \Delta^{\boldsymbol{\rho}_1 \dots \boldsymbol{\rho}_r}_{t_1 \dots t_r}{}_{\lambda_{r+1}\dots\lambda_k} du^{t_1}_{\boldsymbol{\rho}_1} \wedge \dots \wedge du^{t_r}_{\boldsymbol{\sigma}_r} \wedge dx^{\lambda_{r+1}} \wedge \dots \wedge dx^{\lambda_k},$$

where the set $\{{}^{t_1}_{\boldsymbol{\rho}_1} \dots {}^{t_r}_{\boldsymbol{\rho}_r}\}$ is a permutation of the set $\{{}^{i_1}_{\boldsymbol{\sigma}_1} \dots {}^{i_{k-l}}_{\boldsymbol{\sigma}_{k-l}}{}^{j_1}_{\boldsymbol{\tau}_1} \dots {}^{j_l}_{\boldsymbol{\tau}_l}\}$. Then $\sum_i \alpha_{\Delta i} \in \Lambda^k_r$, and its projection on $\mathcal{C}^{k-l}\Lambda^{k-l}_{r,r+1} \wedge \bar{\Lambda}^l_r$ is Δ.

The proof is analogous for $k > n$. QED

We remark that, in general, the above inclusion is a proper inclusion: a sum of elements of the direct summands needs not to be an element of Λ^k_r.

We have a final important consequence of the above results.

COROLLARY 14. *Let $p \leq k$. We have*

$$\mathcal{C}^1\Lambda^k_r = \ker h \quad \text{if} \quad 0 \leq k \leq n, \qquad \mathcal{C}^1\Lambda^k_r = \Lambda^k_r \quad \text{if} \quad k > n.$$

Proof. Let $\alpha \in \Lambda^k_r$. Then, for any section $s : M \to E$ we have $(j_r s)^*\alpha = (j_{r+1}s)^*h(\alpha)$, and $\alpha \in \ker h$ implies $\alpha \in \mathcal{C}^p\Lambda^k_r$. Conversely, suppose $\alpha \in \mathcal{C}^p\Lambda^k_r$. Then we have

$$(j_{r+1}s)^*h(\alpha) = h(\alpha)_{\lambda_1\dots\lambda_k} \circ j_{r+1}s \;\; dx^\lambda_1 \wedge \dots \wedge dx^\lambda_k,$$

hence $h(\alpha) = 0$.

The first assertion comes from the above identities and $\dim M = n$. QED

4 Forms and differential operators

The above construction could be reformulated in a purely algebraic context (see, for example, [12]). One important fact from this theory is the 'parallelism' between the language of forms and the language of differential operators. This allows us to 'import' the theory of adjoint operators and Green's formula in our setting. To do this, we provide a natural isomorphism between the module of contact forms and a space of differential operators.

Let P, Q be modules over an algebra A over $\mathbb{R}$. We recall ([3]) that a *linear differential operator* of order k is defined to be an $\mathbb{R}$-linear map $\Delta : P \to Q$ such that

$$[\delta_{a_0}, [\dots, [\delta_{a_k}, \Delta] \dots]] = 0$$

for all $a_0, \dots, a_k \in A$. Here, square brackets stand for commutators and δ_{a_i} is the multiplication morphism. Of course, linear differential operators of order zero are morphisms of modules. The A-module of differential operators of order k from P to Q is denoted by $\mathrm{Diff}_k(P, Q)$. The A-module of differential operators of any order from P to Q is denoted by $\mathrm{Diff}(P, Q)$. This definition can be generalised to maps between the product of the A-modules P_1,... P_l and Q which are differential operators of order k in each argument, i.e., *multidifferential operators*. The corresponding space is denoted by $\mathrm{Diff}_k(P_1, \dots, P_l; Q)$, or, if $P_1 = \cdots = P_l = P$, by $\mathrm{Diff}_{(l)\,k}(P, Q)$. Accordingly, we define $\mathrm{Diff}_{(l)}(P, Q)$.

When dealing with modules of sections over jets, it is convenient to give a slightly more general definition of differential operator. In fact, we have the natural inclusions $\mathcal{F}_r \subset \mathcal{F}_s$ for $r \leq s$. So, if P is a $\mathcal{F}_r$-module and Q is a $\mathcal{F}_s$ module we can introduce differential operators between P and Q in a natural way. In particular, we consider operators whose expressions contain total derivatives instead of standard ones. More precisely, we say a differential operator $\Delta \colon P \to Q$ (of order k) to be $\mathcal{C}$*-differential* if it can be restricted to the manifolds of the form $j_r(M)$ and $j_s(M)$. In other words, Δ is a $\mathcal{C}$-differential operator if the equality $j_r s(M)^*(\varphi) = 0$, $\varphi \in P$, implies $j_s s(M)^*(\Delta(\varphi)) = 0$ for any section $s\colon M \to E$. In local coordinates, $\mathcal{C}$-differential operators have the form $(a^{\boldsymbol{\sigma}}_{ij} D_{\boldsymbol{\sigma}})$, where $a^{\boldsymbol{\sigma}}_{ij} \in \mathcal{F}_s$, $D_{\boldsymbol{\sigma}} = D_{\sigma_1} \circ \cdots \circ D_{\sigma_k}$.

We denote the $\mathcal{F}_s$-module of $\mathcal{C}$-differential operators of order k from P to Q by $\mathcal{C}\mathrm{Diff}_k(P, Q)$. We also introduce the $\mathcal{F}_s$ module of differential operators from P to Q of any order $\mathcal{C}\mathrm{Diff}(P, Q)$. We can generalize the definition to multi-$\mathcal{C}$-differential operators. In particular, we will be interested to spaces of antisymmetric multi-$\mathcal{C}$-differential operators, which we denote by $\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(l)\,k}(P, Q)$. Analogously, we introduce $\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(l)}(P, Q)$.

Next, we introduce a last important module of vector fields. Namely, let us denote the $\mathcal{F}_r$-module of relative vertical vector field $\varphi\colon J^r\pi \to V\pi$ by $\varkappa_r$. Of course, $\varkappa_0 = \mathcal{V}_0$. Then, any $\varphi \in \varkappa_r$ can be uniquely prolonged to a relative vertical vector field $\ni_\varphi\colon J^{r+s}\pi \to V\pi_s$. It can be proved that $\varkappa$ contains all non-trivial infinitesimal symmetries (even higher order ones) of the Cartan distribution, see [5], for example. If in coordinates $\phi = \phi^i \partial/\partial u^i$, then $\mathrm{Э}_\varphi = D_{\boldsymbol{\sigma}} \varphi^i \partial/\partial u^i_{\boldsymbol{\sigma}}$. Such vector fields are said to be *evolutionary vector fields.*

PROPOSITION 15. *We have the natural isomorphism*

$$\mathcal{C}^p\Lambda^p_{r,r+1} \wedge \bar{\Lambda}^l_r \to \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)\,r}(\varkappa_0, \bar{\Lambda}^l_r), \qquad \alpha \mapsto \nabla_\alpha$$

where $\nabla_\alpha(\varphi_1, \dots, \varphi_p) = \frac{1}{p!} \mathrm{Э}_{\varphi_p} \lrcorner (\dots \lrcorner (\mathrm{Э}_{\varphi_1} \lrcorner \alpha) \dots)$.

The above proposition can be proved by analogy with the infinite order case (see [5]). Just recall that the isomorphism is realized due to the fact that to any vertical tangent vector to $J^r\pi$ there exists an evolutionary field passing through it. If $\alpha \in \mathcal{C}^p\Lambda^p_{r,r+1} \wedge \bar{\Lambda}^l_r$ has the coordinate expression

$$\alpha = \alpha^{\boldsymbol{\sigma}_1 \dots \boldsymbol{\sigma}_p}_{i_1 \dots i_p} \omega^{i_1}_{\boldsymbol{\sigma}_1} \wedge \dots \wedge \omega^{i_p}_{\boldsymbol{\sigma}_p} \wedge \beta,$$

where $\beta \in \bar{\Lambda}^l_r$, then ∇_α has the coordinate expression

$$\nabla_\alpha(\varphi_1, \dots, \varphi_p) = \alpha^{\boldsymbol{\sigma}_1 \dots \boldsymbol{\sigma}_p}_{i_1 \dots i_p} D_{\boldsymbol{\sigma}_1} \varphi^{i_1} \cdots D_{\boldsymbol{\sigma}_p} \varphi^{i_p} \beta.$$

Later on we will use the above isomorphism without an explicit mention.

5 Spectral sequence

The $\mathcal{C}$-spectral sequence has been introduced by Vinogradov in the late Seventies [24, 25, 26]. It is a very powerful tool in the study of differential equations and their symmetries and conservation laws.

Here, we present a new finite order approach to $\mathcal{C}$-spectral sequence on the jet space of order r of a fibred manifold. Such an approach has already been attempted in a particular case [8]. Indeed, the finite order formulation presents some technical difficulties: our main tool is the splitting of theorem 13, where the direct summands have a rather complicated structure with respect to the infinite order analogue.

5.1 Filtration

The module Λ^k_r is filtered by the submodules $\mathcal{C}^p\Lambda^k_r$; namely, we have the obvious *finite* chain of inclusions

$$\Lambda^k_r \stackrel{\mathrm{def}}{=} \mathcal{C}^0\Lambda^k_r \supset \mathcal{C}^1\Lambda^k_r \supset \dots \supset \mathcal{C}^p\Lambda^k_r \supset \dots \supset \mathcal{C}^I\Lambda^k_r \supset \mathcal{C}^{I+1}\Lambda^k_r = \{0\},$$

where I is the dimension of the contact distribution (see [5]).

DEFINITION 16. We say the graded filtration

$$\{\mathcal{C}^p\Lambda^k_r\}_{p\in\mathbb{N}}$$

of Λ^k_r to be the $\mathcal{C}$*-filtration* on the jet space of order r.

The $\mathcal{C}$-filtration gives rise to a spectral sequence in a natural way. The spectral sequences is a well-known tool in Algebraic Topology and Homological Algebra (see, for example, [19]).

DEFINITION 17. We say the spectral sequence $(E_N^{p,q}, e_N)_{N,p,q\in\mathbb{N}}$ (with $p+q=k$) coming from the above filtration to be Vinogradov's *$\mathcal{C}$-spectral sequence of (finite) order r* on the fibred manifold π.

The goal of next subsections is to describe all terms in the spectral sequence that arise from the $\mathcal{C}$-filtration.

5.2 Spectral sequence: the term E_0

As a preliminary step for the study of $(E_0^{p,q}, e_0)$, we look for a description of the spaces $\mathcal{C}^p\Lambda^k_r$. To this aim, we introduce new projections associated to the splitting of proposition 5.

Let $0 \le q \le n$; we denote by H^p the projection

$$\Lambda^{p+q}_{r+1,r} \to \bigoplus_{l=1}^{p} \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p-l)\,r}(\varkappa_0, \mathcal{H}\Lambda^{q+l}_{r+1}); \tag{8}$$

we denote by V^p the complementary projection, i.e., $V^p = \mathrm{id} - H^p$. Of course, $H^p = 0$ if $q = n$ and $H^1 = H$. Also, we denote by h^p and v^p the corresponding restrictions to the subspace Λ^k_r.

REMARK 18. By theorem 13, h^p is not surjective on $\oplus_{l=1}^p \mathcal{C}^{p-l}_{r,r+1} \wedge \bar{\Lambda}^{q+l}_r$ unless $p > 1$ or $q < n-1$ (in these cases there is only 1 summand in (8)).

LEMMA 19. *Let $p \ge 1$. Then, we have*

$$\mathcal{C}^p\Lambda^{p+q}_r \simeq \ker h^p \quad \text{if} \quad q < n;$$
$$\mathcal{C}^p\Lambda^{p+q}_r = \Lambda^{p+q}_r \quad \text{if} \quad q \ge n.$$

Proof. We recall that (corollary 14) the theorem holds for $p = 1$. Then, we have the identities $\ker H^p = \mathrm{im}\, V^p$ and $\mathrm{im}\, V^p = \langle(\mathrm{im}\, V)^p\rangle = \langle(\ker H)^p\rangle$, where $\langle(\mathrm{im}\, V)^p\rangle$ denotes the ideal generated by p-th exterior powers of elements of $\mathrm{im}\, V$ in Λ^{1+q}_r. So, by restriction to Λ^k_r, we have $\ker h^p = \langle(\ker h)^p\rangle$. But, by definition we have $\mathcal{C}^p\Lambda^{p+q}_r = \langle(\ker h)^p\rangle$, hence the result. QED

Now, we compute (E_0, e_0). We recall that $E_0^{p,q} \equiv \mathcal{C}^p\Lambda^{p+q}_r / \mathcal{C}^{p+1}\Lambda^{p+q}_r$. We denote also the differential e_0 (which is the quotient of d) by $\bar{d} \stackrel{\mathrm{def}}{=} e_0$.

LEMMA 20. *We have*

$$E_0^{p,0} = \ker h^p;$$
$$E_0^{p,q} \simeq \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)\,r}(\varkappa_0, \bar{\Lambda}^q_r) \quad \text{if} \quad q \le n;$$
$$E_0^{p,q} = \{0\} \quad \text{otherwise;}$$
$$\bar{d} : E_0^{p,q} \to E_0^{p,q+1} : h^{p+1}(\alpha) \mapsto h^{p+2}(d\alpha).$$

Proof. The first and third assertions are trivial. As for the second one, the inclusion is realized via the injective morphism

$$E_0^{p,q} \equiv \ker h^p \big/ \ker h^{p+1} \to \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)\,r}(\varkappa_0, \bar{\Lambda}^q_r) \colon [\alpha] \mapsto h^{p+1}(\alpha).$$

The above morphism is also surjective: in fact, even if h^{p+1} is not surjective on its target space, it is surjective on each summand of (8).

The differential $\bar{d}$ can be read through the above morphism; we obtain the last assertion. $\boxed{QED}$

PROPOSITION 21. *The bigraded complex (E_0, e_0) is isomorphic to the sequence of complexes*

$$\begin{array}{ccccc}
0 & 0 & 0 & & 0 \\
\uparrow \bar{d} & \uparrow -\bar{d} & \uparrow \bar{d} & & \uparrow (-1)^I \bar{d} \\
\bar{\Lambda}^n_r & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(1)\,r}(\varkappa_0, \bar{\Lambda}^n_r) & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(2)\,r}(\varkappa_0, \bar{\Lambda}^n_r) & \dots & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(I)\,r}(\varkappa_0, \bar{\Lambda}^n_r) \\
\uparrow & \uparrow & \uparrow & & \uparrow \\
\dots & \dots & \dots & \dots & \dots \\
\uparrow \bar{d} & \uparrow -\bar{d} & \uparrow \bar{d} & & \uparrow (-1)^I \bar{d} \\
\bar{\Lambda}^1_r & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(1)\,r}(\varkappa_0, \bar{\Lambda}^1_r) & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(2)\,r}(\varkappa_0, \bar{\Lambda}^1_r) & \dots & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(I)\,r}(\varkappa_0, \bar{\Lambda}^1_r) \\
\uparrow \bar{d} & \uparrow -\bar{d} & \uparrow \bar{d} & & \uparrow (-1)^I \bar{d} \\
\Lambda^0_r & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(1)\,r-1}(\varkappa_0, \mathcal{F}_r) & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(2)\,r-1}(\varkappa_0, \mathcal{F}_r) & \dots & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(I)\,r-1}(\varkappa_0, \mathcal{F}_r) \\
\uparrow & \uparrow & \uparrow & & \uparrow \\
0 & 0 & 0 & \dots & 0
\end{array}$$

The sequence becomes trivial after the I-th column.

The minus signs are put in order to agree with an analogous convention on infinite order variational bicomplexes.

REMARK 22. The differential $\bar{d}$ has the same values as the standard horizontal differential d_H [2, 5, 21] on jet spaces when applied on elements of $E_0^{p,q}$. For a proof in the case $p = 0$, see [2].

5.3 Spectral sequence: the term E_1

In this section we describe the term E_1 of the spectral sequence. We also show that the $\mathcal{C}$-spectral sequence yields an exact sequence of modules which is just the finite order version of the well-known variational sequence.

We recall that $E_1 = H(E_0)$, where the homology is taken with respect to $\bar{d}$. We start by determining the term $E_1^{p,n}$.

THEOREM 23. *We have the diagram*

$$
\begin{array}{ccccccc}
 & 0 & & & 0 & & \\
 & \uparrow & & & \uparrow & & \\
0 \longrightarrow & \bar{\Lambda}^n_r/\bar{d}(\bar{\Lambda}^{n-1}_r) & \xrightarrow{e_1} & \ldots \xrightarrow{e_1} & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)\,r}(\varkappa_0, \bar{\Lambda}^n_r)/\bar{d}(E_0^{p,n-1}) & \xrightarrow{e_1} & \ldots \\
 & \uparrow & & & \uparrow & & \\
 & \bar{\Lambda}^n_r & & \ldots & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)\,r}(\varkappa_0, \bar{\Lambda}^n_r) & & \ldots \\
 & \uparrow{\scriptstyle \bar{d}} & & & \uparrow{\scriptstyle (-1)^I\bar{d}} & & \\
 & \ldots & & & \ldots & & \ldots
\end{array}
$$

where the top row is a complex. The bicomplex is trivial if $p > I$ and vertical arrows with values into the quotients are trivial projections. We have the identifications

$$E_1^{0,n} = \bar{\Lambda}^n_r/\bar{d}(\bar{\Lambda}^{n-1}_r), \tag{9}$$

$$E_1^{p,n} = \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)\,r}(\varkappa_0, \bar{\Lambda}^n_r)/\bar{d}(\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)\,r}(\varkappa_0, \bar{\Lambda}^{n-1}_r), \tag{10}$$

$$e_1^{0,n} : \bar{\Lambda}^n_r/\bar{d}(\bar{\Lambda}^{n-1}_r) \to (E_0^{1,n-1})/\bar{d}(\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(1)\,r}(\varkappa_0, \bar{\Lambda}^{n-1}_r)) \colon$$

$$[h^1(\alpha)] \mapsto [h^2(d\alpha)],$$

$$e_1^{p,n} \colon \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)\,r}(\varkappa_0, \bar{\Lambda}^n_r)/\bar{d}E_0^{p,n-1} \to \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p+1)\,r}(\varkappa_0, \bar{\Lambda}^n_r)/\bar{d}E_0^{p+1,n-1} \colon$$

$$[h^{p+1}(\alpha)] \mapsto [h^{p+2}(d\alpha)].$$

Proof. The above identifications come directly from the definition of E_1. As for the last statement, we have by definition (see, *e. g.*, [19])

$$e_1^{p,1} = \pi \circ \delta,$$

where δ is the Bockstein operator induced by the exact sequence and π is the cohomology map induced by the corresponding map π of the exact sequence. So, suppose that

$$h^{p+1}(\alpha) \in E_0^{p,n} = \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)\,r}(\varkappa_0, \bar{\Lambda}^n_r);$$

we have $\alpha \in \Lambda^{p+n}_r$. Then,

$$\pi(d\alpha) = \bar{d}(\pi(\alpha)) = 0,$$

because $\bar{d}$ raises the degree by 1 on the horizontal factor, so, $d\alpha \in \mathcal{C}^{p+1}\Lambda^{p+1+n}_r$. Being $d(d\alpha) = 0$, $d\alpha$ is closed in $\mathcal{C}^{p+1}\Lambda^{p+1+n}_r$ under the differential d, but $d\alpha$ is not exact in $\mathcal{C}^{p+1}\Lambda^{p+n}_r$, i.e., there does not exist a form $\beta \in \mathcal{C}^{p+1}\Lambda^{p+n}_r = \ker h^{p+1}$ such that $d\beta = \alpha$. Hence, $d\alpha$ determines a cohomology class $[d\alpha]$ in $\mathcal{C}^{p+1}\Lambda^{p+1+n}_r$ which is, by definition, the value of $\delta([h^{p+1}(\alpha)])$. The map π maps $d\alpha$ into $h^{p+2}(d\alpha)$, hence the cohomology class $[d\alpha]$ is mapped into $[h^{p+2}(d\alpha)]$ by π. *QED*

Now we determine $E_1^{p,q}$. We need some important preliminary results.

We observe that the $\mathcal{C}$-spectral sequence of order r converges to the de Rham cohomology of E. This is due to the fact that the $\mathcal{C}$-spectral sequence is a first quadrant spectral sequence. So, according to the standard definition of convergence [6, 19], there exists $n_0 \in \mathbb{N}$ such that $E_{n_0} = E_s$ for $s > n_0$, and E_{n_0} is isomorphic to the quotient vector spaces of the filtration

$$H^*(\Lambda^*) \supset iH^*(\mathcal{C}^1\Lambda^*) \supset i^2 H^*(\mathcal{C}^2\Lambda^*) \supset \cdots \supset i^I H^*(\mathcal{C}^I\Lambda^*) \supset 0,$$

of the de Rham cohomology of E (i is the natural inclusion).

LEMMA 24. *The sequence*

$$0 \longrightarrow \mathcal{C}^p\Lambda_r^p \xrightarrow{d} \mathcal{C}^p\Lambda_r^{p+1} \xrightarrow{d} \dots \xrightarrow{d} \mathcal{C}^p\Lambda_r^{p+n-1} \xrightarrow{d} \dots$$

is exact up to the term $\mathcal{C}^p\Lambda_r^{p+n-1}$.

Proof. We generalize arguments and computations from [15]. The modules $\mathcal{C}^p\Lambda_r^k$ are the spaces of global sections of the corresponding sheaves of contact forms. Such sheaves are soft sheaves because they are sheaves of modules over a sheaf of rings, $\mathcal{F}_r$, which admit a partition of unity. We want to prove that the corresponding sheaf sequence is exact up the term $\mathcal{C}^p\Lambda_r^{p+n-1}$. In this case, such a sequence would be a soft resolution of $\mathcal{C}^p\Lambda_r^p$, hence acyclic.

First, we prove exactness at $\mathcal{C}^p\Lambda_r^p$. We proceed by induction on r. Let $\alpha \in \mathcal{C}^p\Lambda_r^p$ such that $d\alpha = 0$. If we have the coordinate expression $\alpha = \alpha_{i_1\dots i_p}^{\boldsymbol{\sigma}_1\dots\boldsymbol{\sigma}_p}\,\omega_{\boldsymbol{\sigma}_1}^{i_1} \wedge \cdots \wedge \omega_{\boldsymbol{\sigma}_p}^{i_p}$, where $|\boldsymbol{\sigma}_k| \leq r-1$, then

$$d\alpha = d\alpha_{i_1\dots i_p}^{\boldsymbol{\sigma}_1\dots\boldsymbol{\sigma}_p} \wedge \omega_{\boldsymbol{\sigma}_1}^{i_1} \wedge \cdots \wedge \omega_{\boldsymbol{\sigma}_p}^{i_p} + \alpha_{i_1\dots i_p}^{\boldsymbol{\sigma}_1\dots\boldsymbol{\sigma}_p}\omega_{\boldsymbol{\sigma}_1}^{i_1} \wedge \cdots \wedge dx^\lambda \wedge \omega_{\boldsymbol{\sigma}_k\lambda}^{i_k} \wedge \cdots \wedge \omega_{\boldsymbol{\sigma}_p}^{i_p} \tag{11}$$

where $1 \leq k \leq p$. Hence $\alpha_{i_1\dots i_p}^{\boldsymbol{\sigma}_1\dots\boldsymbol{\sigma}_p} = 0$ if $|\boldsymbol{\sigma}_k| = r-1$ for some k. The induction yields $\alpha = 0$.

Now, let $\alpha \in \mathcal{C}^p\Lambda_r^h$, with $h > p$. We recall the *contact homotopy operator* [15]. Namely, let $(x^\lambda, y_{\boldsymbol{\sigma}}^i)$ be a fibred chart on $J^r\pi$. We define the map $H(t, x^\lambda, u_{\boldsymbol{\sigma}}^i) = (x^\lambda, tu_{\boldsymbol{\sigma}}^i)$, $0 \leq t \leq 1$. We observe that $H^*\omega_{\boldsymbol{\sigma}}^i = t\omega_{\boldsymbol{\sigma}}^i + u_{\boldsymbol{\sigma}}^i dt$. Moreover, $H^*\alpha = \alpha' \wedge dt + \alpha''$. So, we define the contact homotopy operator to be the map

$$A\alpha \stackrel{\text{def}}{=} \int_0^1 \alpha' \wedge dt.$$

It is easy to check that $\alpha = Ad\alpha + dA\alpha$. Now, let $d\alpha = 0$. The proof is complete if we show that $A\alpha$ is a p-contact form. Unfortunately, the properties of H^* imply that $A\alpha$ is in general a $(p-1)$-contact form. But $d\alpha = 0$, hence we can prove that $\alpha = \beta + d\gamma$, where β is $(p+1)$-contact and γ is p-contact. We have the coordinate expression

$$\alpha = A_{i_1\dots i_p}^{\boldsymbol{\sigma}_1\dots\boldsymbol{\sigma}_p} \wedge \omega_{\boldsymbol{\sigma}_1}^{i_1} \wedge \cdots \wedge \omega_{\boldsymbol{\sigma}_p}^{i_p} + d(B_{i_1\dots i_p}^{\boldsymbol{\sigma}_1\dots\boldsymbol{\sigma}_p} \wedge \omega_{\boldsymbol{\sigma}_1}^{i_1} \wedge \cdots \wedge \omega_{\boldsymbol{\sigma}_p}^{i_p})$$

where $A^{\cdots}_{\cdots}, B^{\cdots}_{\cdots} \in \Lambda^{h-p}_r$ and $|\boldsymbol{\sigma}_k| \leq r-1$. Then the coordinate expression of $d\alpha$ is similar to (11). Split $A^{\cdots}_{\cdots} = A^{\cdots}_{\cdots h} + A^{\cdots}_{\cdots c}$, where the first summand is an horizontal form and the second is a contact form. Then by induction on r it is easy to see that $A^{\cdots}_{\cdots h} = 0$. Setting

$$\beta \stackrel{\text{def}}{=} A^{\boldsymbol{\sigma}_1 \ldots \boldsymbol{\sigma}_p}_{i_1 \ldots i_p \; c} \wedge \omega^{i_1}_{\boldsymbol{\sigma}_1} \wedge \cdots \wedge \omega^{i_p}_{\boldsymbol{\sigma}_p}$$

we have $\alpha = \beta + d\gamma$, $d\beta = 0$, hence $\alpha = d(A\beta + \gamma)$. QED

The above result allows us to compute the term $E^{p,q}_1$.

THEOREM 25. *We have*

1. $E^{0,q}_1 = H^q(E)$, *for* $q \neq n$;
2. $E^{p,n}_1 = H^p(E)$, *for* $p \geq 1$;
3. $E^{p,q}_1 = 0$ *for* $q \neq n$ *and* $p \neq 0$.

Proof. The first result follows from the fact that $E^{0,q}_0$ is the quotient of the de Rham sequence with an exact sequence (see the above lemma), hence its cohomology is the de Rham cohomology of $J^r\pi$. This latter cohomology is equal to $H^*(E)$ because $J^r\pi$ has topologically trivial fibre over E.

The third statement is a direct consequence of the above lemma

The second statement comes from a straightforward computation and the convergence of the $\mathcal{C}$-spectral sequence to the de Rham cohomology. QED

5.4 Spectral sequence: the variational sequence

The results of theorem 25 can be used to produce a new sequence which is of fundamental importance, namely the variational sequence. The spaces of the sequence are cohomology classes of E_0 and E_1. In order to give an explicit expression to classes in E_1 a key role will be played by the intrinsic definition of adjoint operator [5]

THEOREM 26. *We have the complex*

$$\ldots \xrightarrow{\bar{d}} \bar{\Lambda}^n_r \xrightarrow{\tilde{e}_1} \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(1)\,r}(\varkappa_0, \bar{\Lambda}^n_r) / \bar{d}(\mathcal{C}\mathrm{Diff}1r(\varkappa_0, \Lambda^{n-1}_r)) \xrightarrow{e_1} \ldots,$$

where $\tilde{e}_1$ *is the map which make the following diagram commute*

$$\begin{array}{ccc} \bar{\Lambda}^n_r & \xrightarrow{\tilde{e}_1} & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(1)\,r}(\varkappa_0, \bar{\Lambda}^n_r) / \bar{d}(\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(1)\,r}(\varkappa_0, \bar{\Lambda}^{n-1}_r)) \\ & \searrow \qquad \nearrow_{e_1} & \\ & \bar{\Lambda}^n_r / \bar{d}(\bar{\Lambda}^{n-1}_r) & \end{array}$$

The cohomology of the above complex turns out to be naturally isomorphic to the de Rham cohomology of E.

DEFINITION 27. We say the above complex to be the *finite order $\mathcal{C}$-variational sequence associated with the $\mathcal{C}$-spectral sequence of order r on $\pi\colon E \to M$.*

The word 'variational' comes from the fact that we can identify the objects of the space $\bar{\Lambda}^n_r$ with $(r+1)$*-st order Lagrangians* [31, 32]. Moreover, next two spaces in the sequence can be identified with a space of (finite order) *Euler–Lagrange morphisms* and a space of (finite order) *Helmholtz morphism*, and the differential e_1 is the operator sending Lagrangians into corresponding Euler–Lagrange morphism and Euler–Lagrange type morphisms into Helmholtz morphisms.

The $\mathcal{C}$-variational sequence is defined through some quotient spaces. Now, we prove that each equivalence class in these spaces can be represented by a distinguished form.

To this aim, we observe that pull-back allows to take $\bar{d} = \widehat{d}$. More precisely, we can consider $\alpha \in \bar{\Lambda}^q_r \subset \mathcal{H}\Lambda^q_{r+1}$, so that the horizontalization on $(r+1)$-st order jets is the identity on α. In this way, the complementary map v fulfills $v(\alpha) = 0$ on the $(r+1)$-st order jet. Hence $\bar{d}(\alpha) = \widehat{d}(\alpha)$. Also, it is easily shown [32] that the r-th order $\mathcal{C}$-variational sequence is embedded via pull-back into the $(r+1)$-st order one. More generally, it can be proved that the direct limit of the r-th order $\mathcal{C}$-variational sequence is just the standard infinite order $\mathcal{C}$-variational sequence (see also next subsection). Hence, we have the embedding

$$\begin{aligned}(12)\qquad \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)\,r}(\varkappa_0, \bar{\Lambda}^n_r)/\bar{d}(\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)\,r}(\varkappa_0, \bar{\Lambda}^{n-1}_r)) \hookrightarrow \\ \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa, \bar{\Lambda}^n)/\bar{d}(\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa, \bar{\Lambda}^{n-1})),\end{aligned}$$

where $\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}$ is the space of operators of any order, $\bar{\Lambda}^k$ is the space of horizontal forms on the jet space of any order, and $\varkappa$ is the space of relative vertical vector fields of any order.

Let $\mathcal{F}$ be the space of functions on jet spaces of any order, and set $\widehat{\varkappa} \overset{\mathrm{def}}{=} \mathrm{Hom}_{\mathcal{F}}(\varkappa, \bar{\Lambda}^n)$. We recall that, if $\Delta\colon P \to Q$ is a $\mathcal{C}$-differential operator, then [5] there exists an operator $\Delta^*\colon \widehat{Q} \to \widehat{P}$. It fulfills

$$(13)\qquad \widehat{q}(\Delta(p)) - (\Delta^*(\widehat{q}))(p) = \widehat{d}\omega_{p,\widehat{q}}(\Delta).$$

In coordinates, if $\Delta = \Delta^{\boldsymbol{\sigma}}_{ij} D_{\boldsymbol{\sigma}}$, then

$$\Delta^* = (-1)^{|\boldsymbol{\sigma}|} D_{\boldsymbol{\sigma}} \circ \Delta^{\boldsymbol{\sigma}}_{ji}.$$

Now, the following well-known isomorphism holds ([24, 25, 26]; see also [5, p. 192])

$$(14)\qquad \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa, \bar{\Lambda}^n)/\bar{d}(\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa, \bar{\Lambda}^{n-1})) \simeq K_p(\varkappa),$$

where $K_p(\varkappa) \subset \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(l-1)}(\varkappa, \widehat{\varkappa})$ is the subspace of operators ∇ which are skew-adjoint in each argument, i.e.,

$$(15)\qquad (\nabla(\varphi_1, \dots, \varphi_{p-2}))^* = -\nabla(\varphi_1, \dots, \varphi_{p-2})$$

for all $\varphi_1, \ldots, \varphi_{l-2} \in \varkappa$. Note that, if $p = 1$, then the isomorphism reads as the evaluation of the adjoint of the given operator at the constant function 1 [5].

The above considerations show that the equivalence class $[\alpha]$ in the quotient space with contact degree p is represented through the embedding (12) and the isomorphism (14) as the differential operator $\bar{\nabla}_\alpha$ obtained after skew-adjoining α in its first $(p-1)$-arguments and adjoining it in its p-th argument.

Accordingly, the space of distinguished representatives of quotient spaces of order r is the subspace of $K_p(\varkappa)$ made by $(2r+1)$-st order operators of the form of (15).

Let us give a look in coordinates: $\alpha \in \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(1)\,r}(\varkappa_0, \bar{\Lambda}^n_r)$ has the expression

$$\alpha = \alpha^{\boldsymbol{\sigma}_1 \ldots \boldsymbol{\sigma}_{p-1} \boldsymbol{\tau}}_{i_1 \ldots i_{p-1} j} \omega^{i_1}_{\boldsymbol{\sigma}_1} \wedge \cdots \wedge \omega^{i_{p-1}}_{\boldsymbol{\sigma}_{p-1}} \wedge \omega^j_{\boldsymbol{\tau}} \wedge \mathrm{Vol}_M,$$

where $\mathrm{Vol}_M \overset{\mathrm{def}}{=} dx^1 \wedge \cdots \wedge dx^n$ is the local volume on M. Hence, if $p = 1$, then

$$\bar{\nabla}_\alpha = (-1)^{|\boldsymbol{\sigma}|} D_{\boldsymbol{\sigma}} \alpha^{\boldsymbol{\sigma}}_i \omega^i \wedge \mathrm{Vol}_M \,.$$

This clearly shows that the first quotient space in the variational sequence is the space of Euler–Lagrange type operators. If $p = 2$, then equations (13), (15) yield

$$\begin{aligned} \bar{\nabla}_\alpha(\varphi_1)(\varphi_2) &= (-1)^{|\boldsymbol{\tau}|} D_{\boldsymbol{\tau}} \left(\alpha^{\boldsymbol{\sigma}_1 \boldsymbol{\tau}}_{i_1\ j} D_{\boldsymbol{\sigma}_1} \varphi^{i_1}_1 \right) \varphi^j_2 \,\mathrm{Vol}_M \\ &= \sum_{0 \le |\boldsymbol{\nu}| + |\boldsymbol{\tau}_1| \le r} (-1)^{|(\boldsymbol{\nu},\boldsymbol{\tau}_1)|} \frac{(\boldsymbol{\nu},\boldsymbol{\tau}_1)!}{\boldsymbol{\nu}! \boldsymbol{\tau}_1!} D_{\boldsymbol{\nu}} \alpha^{\boldsymbol{\sigma}_1 (\boldsymbol{\nu},\boldsymbol{\tau}_1)}_{i_1\ j} D_{(\boldsymbol{\tau}_1,\boldsymbol{\sigma}_1)} \varphi^{i_1}_1 \varphi^j_2 \,\mathrm{Vol}_M \\ &= \left(\sum_{\substack{(\boldsymbol{\sigma}_1,\boldsymbol{\tau}_1) = \boldsymbol{\rho}_1 \\ 0 \le |\boldsymbol{\rho}_1| \le 2r}} (-1)^{|(\boldsymbol{\nu},\boldsymbol{\tau}_1)|} \frac{(\boldsymbol{\nu},\boldsymbol{\tau}_1)!}{\boldsymbol{\nu}! \boldsymbol{\tau}_1!} D_{\boldsymbol{\nu}} \alpha^{\boldsymbol{\sigma}_1 (\boldsymbol{\nu},\boldsymbol{\tau}_1)}_{i_1\ j} \right) D_{\boldsymbol{\rho}_1} \varphi^{i_1}_1 \varphi^j_2 \,\mathrm{Vol}_M, \end{aligned} \tag{16}$$

where $(\boldsymbol{\sigma}, \boldsymbol{\rho})$ denotes the union of the multiindexes. Note that we also used the Leibniz rule for total derivatives [21].

This clearly shows that the second quotient space in the variational sequence is the space of Helmholtz type operators. We recall that the Helmholtz operator of an Euler–Lagrange type operator η is just $e_1(\eta)$. If $e_1(\eta) = 0$ then the local exactness of the variational sequence tells us that η is (locally) the Euler–Lagrange operator of a Lagrangian.

Through the above expressions it is possible to derive a representation formula for any p.

REMARK 28. The expression of e_1 between quotient spaces can be derived by observing that e_1 is the quotient of the contact (or vertical) differential $\mathcal{C}d$ (also denoted by d_V, [31, 32]). Namely, $e_1([\alpha]) = [\mathcal{C}d(\alpha)]$.

REMARK 29. We could consider the 'complementary' problem to the representative's one. More precisely, given $\alpha \in \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)\,r}(\varkappa_0, \bar{\Lambda}^n_r)$ we can look for a section q fulfilling $\alpha = \nabla_\alpha + \bar{d}q$. Such a section always exists due to the vanishing of the cohomology of $\bar{d}$ on the space where q lives.

This problem, for $p = 1$, is the search for a Poincaré–Cartan form (see [11, 31], for example). Alonso Blanco [1] is able to determine one section q through a connection for any p.

5.5 Comparisons

We can perform two different kinds of comparisons.

Comparison with the standard infinite order $\mathcal{C}$-spectral sequence. The analysis has been done in [32]. Here we summarize the main steps.

It is clear that pull-back provides an inclusion of the r-th order $\mathcal{C}$-spectral sequence into the $(r+1)$-st order $\mathcal{C}$-spectral sequence. This yields a sequence of spectral sequences whose direct limit is the infinite order $\mathcal{C}$-spectral sequence [32]. By the way, it can be easily proved (see remark 22) that the direct limit of $\bar{d}$ and $\widehat{d}$ is the same.

One of the main differences between the finite order approach and the infinite order approach is that in the infinite order case the diagram (23) is a bicomplex, where horizontal arrows are provided by $\mathcal{C}d$ (see remark 22). In particular, it is easily proved that the differential e_1 turns out to be the equivalence class of the differential $\mathcal{C}d$. In the finite order case this differential does not provide any additional complex because it raises the order of jet by one.

Another important point is that in the finite order approach there are only finitely many non-zero $E_N^{p,q}$, because de Rham complex on finite order jets ‘stops’.

In the finite order approach we recover some well-known results of the infinite order case but proofs are slightly different. This can give some new insight in the theory.

Comparison with Krupka's finite order variational sequence. Krupka's sequence [15] is obtained by quotienting the de Rham sequence on $J^r\pi$ with a natural subsequence. The first n terms of this subsequence are given in lemma 24, next ones are the spaces of sections which are *locally* in the space $\ker h^p + d\ker h^{p-1}$. In [31, 32] it is proved that Krupka's sequence is isomorphic to the sequence

$$\dots \xrightarrow{\mathcal{E}_{n-1}} \bar{\Lambda}^n_r \xrightarrow{\mathcal{E}_n} \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(1)\,r}(\varkappa_0, \bar{\Lambda}^n_r)/h^2(d\ker h^1)^s \xrightarrow{\mathcal{E}_{n+1}}$$

$$\dots \xrightarrow{\mathcal{E}_{n+i-1}} \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(i)\,r}(\varkappa_0, \bar{\Lambda}^n_r)/h^{i+1}(d\ker h^i)^s \xrightarrow{\mathcal{E}_{n+i}} \dots$$

where $\mathcal{E}_k([h^p(\alpha)]) = [h^{p+1}(d\alpha)]$, and $h^{i+1}(d\ker h^i)^s$ stands for the space of sections which are *locally* in the space $h^{i+1}(d\ker h^i)$. Moreover, in [32] it is proved that the above sequence and the finite order $\mathcal{C}$-variational sequence are isomorphic up to the $(n+1)$-st term.

We can easily complete the proof and show the equivalence of both finite order sequences in all their terms. In fact,

$$h^{i+1}(d\ker h^i)^s = h^{i+1}(d\ker h^i),$$

because the cohomology of $e_0 = \bar{d}$ vanish (see theorems 23 and 25).

In particular, the second quotient space of the $\mathcal{C}$-variational sequence is naturally isomorphic to the $(n+2)$-th space of Krupka's variational sequence. This means that we provided in equation (16) a way to represent all elements of degree $(n+2)$ of Krupka's variational sequence, and the way to a possible generalization to $(n+p)$.

Comparison with Anderson and Duchamp's approach. There is only one more approach to finite order variational sequences [2], but the sequence provided in that paper stops after the $(n+1)$-st term. Moreover, this sequence has been derived directly from coordinate computations. Also, from the coordinate expression it is easy to see that the sequence coincides with the $\mathcal{C}$-variational sequence up to the n-th term. The last map in Anderson and Duchamp's paper has the same value of the corresponding map in the $\mathcal{C}$-variational sequence but is defined between slightly different spaces.

To the author's knowledge the $\mathcal{C}$-variational sequence, Krupka's variational sequence and Anderson and Duchamp's sequence are the only finite order formulation of variational sequences.

6 Conclusions

We derived a new finite order formulation of Vinogradov's $\mathcal{C}$-spectral sequence. We recovered most results of the infinite order formulations in the finite order case. We have shown that the associated $\mathcal{C}$-variational sequence is isomorphic to Krupka's one. But the techniques employed throughout this paper can be easily generalized, e.g., to jets of submanifolds and differential equations, along the lines of similar developments in the infinite order case. Moreover, we have briefly seen how the infinite order scheme can be reached from the finite order one through a direct limit process.

It seems to be clear that working on finite order jets produces almost the same amount of information as working on infinite jets, from a 'structural' viewpoint. But there is a cost in carrying the order of jets in all computations.

The author's opinion is that it is possible to skip problems related to the order when the target of the research does not involve it. Actually, in this paper we have shown that there exists the possibility of computing orders also in the infinite order $\mathcal{C}$-spectral sequence. It is not necessary to use such a possibility from the very beginning of any investigation: in fact, the order can be computed at any step. Of course, there are problems where the order is fundamental. Main examples are the problem of the minimal order variational potential for a variationally trivial Lagrangian and the problem of the minimal order Lagrangian for a given variationally trivial Euler–Lagrange operator [10, 16].

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