Variational sequences

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Abstract

Variational sequences are complexes of modules or sheaf sequences in which one of the operations is the Euler–Lagrange operator, *i.e.*, the differential operator taking a Lagrangian into its Euler–Lagrange form, whose kernel is the Euler–Lagrange equation.

In this paper we present the most common approaches to variational sequences and discuss some directions of the current research on the topic.

Key words: Jet spaces, variational sequence, variational bicomplex, $\mathcal{C}$-spectral sequence, horizontal cohomology, characteristic cohomology.

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Contents

Introduction

The modern differential geometric approach to mechanics and field theory inspired many scientists coming from different areas of mathematics and theoretical physics to the development of a differential geometric theory of the calculus of variations (see, for example, [40, 45, 69, 108]). Relevant objects of the calculus of variations (like Lagrangians, components of Euler–Lagrange equations) were interpreted as differential forms on jet spaces[1] of a fibred manifold.

Fibred manifolds where chosen to provide a geometric model of the space of independent and dependent coordinates. This is not the most general model, see subsection 6.2.

Soon it was realized that operations like the one of passing from a Lagrangian to its Euler–Lagrange form were part of a complex, namely, the *variational sequence*. The foundational contributions to variational sequences (and much more) can be found in the papers [5, 22, 29, 30, 42, 55, 70, 89, 102, 106, 109, 110, 115, 116, 118]. More details on the development of the subject can be found in section 7.

The variational sequence is a tool that allows to fit several important problems of the calculus of variations all at once. Let us describe two of the most important among such problems.

- Given a set of Euler–Lagrange equations, the vanishing of Helmholtz conditions is a necessary and sufficient condition for the existence of a local Lagrangian for the given equations. (see, *e.g.*, [24, 105] for more details on Helmholtz conditions). What about the domain of definition of the Lagrangian? Does there exist a global Lagrangian? This problem is said to be the *inverse problem* of the calculus of variations, despite the fact that it is not the only inverse problem that can be considered in this framework.

- It is important to be able to determine all variationally trivial Lagrangians, depending on derivatives of a prescribed order, defined on a given fibred manifold. Those are Lagrangians whose Euler–Lagrange equation identically vanish. For example, in liquid crystals theories [36] minima of the action functional can be computed by adding to the physical Lagrangian a trivial Lagrangian. Such trivial Lagrangians are known to be locally of the type of a 'total divergence' of an $n-1$-form. But what about their dependence on highest order derivatives? Moreover, another inverse problem arises: are they global total divergences

[1]The reader is invited to see the paper [100] of this Handbook about jet spaces.

or not?

Let us see what are the answers of variational sequence theories to the above problems in an intuitive way. Let us denote by C the space of currents[2], by L the space of Lagrangian forms, by E the space of Euler–Lagrange-type forms and by H the space of Helmholtz forms. Then, the variational sequence is a sequence of modules (or of sheaves, depending on the approaches) of the type

$$\cdots \longrightarrow C \xrightarrow{d_H} L \xrightarrow{\mathcal{E}} E \xrightarrow{\mathcal{H}} H \xrightarrow{\mathcal{D}} \cdots \qquad ,$$

where d_H is the operator of total divergence, $\mathcal{E}$ is the operator that takes a Lagrangian into the corresponding Euler–Lagrange form, $\mathcal{H}$ is the operator which takes an Euler–Lagrange type form into the corresponding Helmholtz form, and $\mathcal{D}$ is a further operator of the complex.

The repeated application of two consecutive operators of the sequence is identically zero: this is why the homological algebra term 'complex' is used for the above sequence. In the theory of variational sequences the following facts are proved about the previous problems.

- Let $\eta \in E$ be a Euler–Lagrange form. Then $\eta = \mathcal{E}(\lambda)$ for a locally defined Lagrangian $\lambda \in L$ if and only if $\mathcal{H}(\eta) = 0$. The space $\ker \mathcal{H}/\operatorname{im}\mathcal{E}$ is isomorphic to the $n+1$-st de Rham cohomology of the total space of the fibred manifold. This is the solution of the so-called *global inverse problem.*

- The set of variationally trivial Lagrangians is $\ker \mathcal{E}$. The space $\ker \mathcal{E}/\operatorname{im} d_H$ is isomorphic to the n-th cohomology class of the total space of the fibred manifold. This enables us to compute which variationally trivial Lagrangians are of the global or local form of a total divergence. More information on the structure of such Lagrangians can be found in section 6.1.

Now, let us describe the structure of the paper.

In section 1 some basic facts on jet spaces are recalled. The interested reader may consult [1, 18, 79, 82, 91, 98] and the paper [100] in this Handbook for detailed introductions to jets.

[2]Here 'currents' are $n-1$-forms, hence they can be integrated on $n-1$-dimensional submanifolds. This includes conserved quantities (or conserved currents). The term 'currents' from classical calculus of variations admits a modern generalization [44] which is not dealt with hereby.

Next section is devoted to contact forms, which are forms whose pull-back by any section of the fibred manifold identically vanishes. The horizontalization is introduced in order to be able to single out the part of a form whose pull-back by any section does not identically vanish. In order to overcome some technical difficulties, infinite order jets are introduced. In practice, this trick amount at dealing with forms which are defined on an arbitrary, but finite, order jet space.

Section 3 contains a description of one of the approaches to variational sequences on fibred manifolds: the variational bicomplex. This approach has been developed mostly in [102, 109, 110, 111]. Local exactness and global cohomological properties of the variational bicomplex are discussed.

In section 4 another important approach to variational sequences is presented: the $\mathcal{C}$-spectral sequence approach by [115, 116, 118]. Contact forms provide a differential filtration of the space of forms on jets. This filtration induces a spectral sequence, the $\mathcal{C}$-spectral sequence, in a standard way. A part of the variational bicomplex and the variational sequence is recovered as some of the differential groups in the $\mathcal{C}$-spectral sequence. The $\mathcal{C}$-spectral sequence also yields a variational sequence on manifolds without fibrings (see subsection 6.2) and on differential equations (see subsection 6.3). In particular, in the latter case, the $\mathcal{C}$-spectral sequence yields differential and topological invariants of the equation, among which there are conservation laws (this particular aspect received foundational contributions also by [22, 23, 106]).

The above approaches were formulated on infinite order jets. In [5, 34, 70] a finite order variational sequence on jets of fibred manifolds is proposed. The approach in [70] is described in section 5, together with comparisons with the above infinite order approaches.

Unfortunately, space constraints do not allow to write a complete text on variational sequences. For this reason, while foundations of the theory are exposed in the above sections in the most possible detailed way (but without detailed proofs), in section 6 there is a collection of references to many interesting theoretical and applied topics like the equivariant inverse problem, symmetries of variational forms, variational sequences on jets of submanifolds, etc.. The reader who is interested in more detailed foundational expositions of the subject could consult the following books.

[4] This book is unpublished, yet it is a good source of examples, calculations and results which never appeared elsewhere.

[14] The book is devoted to the inverse problem in mechanics (one independent variable).

[23] The book covers some geometric aspects of the calculus of variations which are quite close to those of this paper, but in the framework of exterior differential systems.

[18] It is a book on the geometry of differential equations, with one chapter devoted to the variational sequence on jets of fibrings and on differential equations, with a focus on symmetries and conservation laws.

[32] *Idem.* There is one section about the variational bicomplex.

[57] *Idem.* The formalism is quite close to that of [4].

[66] *Idem.* The formalism is the same as in [18] but with a lot more of theoretical material, like the k-lines theorem.

[91] *Idem.* There is also a section on variational multivectors, which are dual objects of variational forms (see subsection 6.5).

[98] The book deals with jets of fibrings and has a final chapter on the variational sequence on infinite order jets.

[121] It is a book on the geometry of differential equations and the $\mathcal{C}$-spectral sequence (see section 4), with a mostly theoretical exposition.

[127] *Idem*, but there are some examples and applications.

We also stress that two very good web sites for this topic are the web site at Utah State University of Logan `http://www.math.usu.edu/~fg_mp` (which seems to be no longer actively maintained) and the 'diffiety' web site `http://diffiety.ac.ru`.

The paper ends with some notes on the development of the subject and a relatively complete bibliography. Despite the fact that we did extensive bibliographical researches the subject is quite vast and it is possible that some issues have been forgotten or not properly mentioned. For this reason, we excuse ourselves in advance with the scientists whose contribution was hereby overlooked or misunderstood.

As a last comment, we observe that we had to make a synthesis from a lot of sources. For this reason we adopted notation that did not come from a single source, but has the advantage of being able to express all approaches at once.

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1 Preliminaries

Manifolds and maps between manifolds are C^∞. All morphisms of fibred manifolds (and hence bundles) will be morphisms over the identity of the base manifold, unless otherwise specified. In particular, when speaking of 'forms' we will always mean 'C^∞ differential forms'.

Some parts of this paper deals with sheaves. A concise but fairly complete exposition of sheaf theory can be found in [126]; it covers all the needs of our exposition.

Now, we recall some basic facts on jet spaces. Our framework is a fibred manifold

$$\pi : E \to M,$$

with $\dim M = n$, $\dim E = n + m$, n, $m \geq 1$. We have the vector subbundle $VE \stackrel{\text{def}}{=} \ker T\pi$ of TE, which is made by vectors which are tangent to the fibres of E.

For $1 \leq r$, we are concerned with the r-th jet space $J^r\pi$; we also set $J^0\pi \equiv E$. For $0 \leq s < r$ we recall the natural fibrings

$$\pi_{r,s} : J^r\pi \to J^s\pi, \qquad \pi_r : J^r\pi \to M,$$

and the affine bundle $\pi_{r,r-1} : J^r\pi \to J^{r-1}\pi$ associated with the vector bundle $\odot^r T^*M \otimes_{J^{r-1}\pi} VE \to J^{r-1}\pi$.

Charts on E adapted to the fibring are denoted by (x^λ, u^i). Greek indices λ, μ,... run from 1 to n and label base coordinates, Latin indices i, j,... run from 1 to m and label fibre coordinates, unless otherwise specified. We denote by $(\partial/\partial x^\lambda, \partial/\partial u^i)$ and (dx^λ, du^i), respectively, the local bases of vector fields and 1-forms on E induced by an adapted chart.

Multiindices are needed in order to label derivative coordinates on jet spaces. It is possible to use general multiindices or symmetrized multiindices in order to label derivatives. There are advantages and disadvantages of both approaches; to the purposes of this paper we prefer to use the symmetrized multiindices because of the one-to-one correspondence between them and coordinates on jets. In particular, following the approach of [98] we denote multi-indices by boldface Greek letters such as $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n) \in \mathbb{N}^n$. We also set $|\boldsymbol{\sigma}| \overset{\text{def}}{=} \sum_i \sigma_i$ and $\boldsymbol{\sigma}! \overset{\text{def}}{=} \sigma_1! \cdots \sigma_n!$. Multiindices can be summed in an obvious way; the sum of a multiindex with a Greek index $\boldsymbol{\sigma} + \lambda$ will denote the sum of $\boldsymbol{\sigma}$ and the multiindex $(0, \dots, 1, 0, \dots, 0)$, where 1 is at the λ-th entry.

The charts induced on $J^r\pi$ are denoted by $(x^\lambda, u^i_{\boldsymbol{\sigma}})$, where $0 \leq |\boldsymbol{\sigma}| \leq r$ and $u^i_0 \overset{\text{def}}{=} u^i$. The local vector fields and forms of $J^r\pi$ induced by the fibre coordinates are denoted by $(\partial/\partial u^i_{\boldsymbol{\sigma}})$ and $(du^i_{\boldsymbol{\sigma}})$, $0 \leq |\boldsymbol{\sigma}| \leq r, 1 \leq i \leq m$, respectively.

An r-th order (ordinary or partial) *differential equation* is, by definition, a submanifold $Y \subset J^r\pi$.

We denote by $j_r s\colon M \to J^r\pi$ the jet prolongation of a section $s\colon M \to E$ and by $J^r f\colon J^r\pi \to J^r\pi$ the jet prolongation of a fibred morphism $f\colon E \to E$ over the identity. Any vector field $X\colon E \to TE$ which projects onto a vector field $X\colon M \to TM$ can be prolonged to a vector field $X^r\colon J^r\pi \to TJ^r\pi$ by prolonging its flow; its coordinate expression is well-known (see, *e.g.*, [18, 91, 98]).

The fundamental geometric structure on jets is the *contact distribution*, or *Cartan distribution*, $C^r \subset TJ^r\pi$. It is the distribution on $J^r\pi$ generated by all vectors which are tangent to the image $j_r s(M) \subset J^r\pi$ of a prolonged section $j_r s$. It is locally generated by the vector fields

$$D_\lambda = \frac{\partial}{\partial x^\lambda} + u^i_{\boldsymbol{\sigma}\lambda} \frac{\partial}{\partial u^i_{\boldsymbol{\sigma}}}, \qquad \frac{\partial}{\partial u^i_{\boldsymbol{\tau}}}, \tag{1}$$

with $0 \leq |\boldsymbol{\sigma}| \leq r - 1$, $|\boldsymbol{\tau}| = r$.

Remark 1.1. The contact distribution on finite order jets is not involutive. Indeed, despite the fact that $[D_\lambda, D_\mu] = 0$, if $\boldsymbol{\tau} = \boldsymbol{\sigma} + \lambda$ then $[D_\lambda, \partial/\partial u^i_{\boldsymbol{\tau}}] = -\partial/\partial u^i_{\boldsymbol{\sigma}}$. Moreover, the contact distribution on finite order jets does not admit a natural direct summand that complement it to $TJ^r\pi$. The above two facts are the main motivation to the passage to infinite order jets in order to formulate the variational sequence.

While the contact distribution has an essential importance in the symmetry analysis of PDE [18, 91], in this context the dual concept of contact differential form will play a central role.

2 Contact forms

2.1 Contact forms

Let us denote by $\mathcal{F}_r = C^\infty(J^r\pi)$ the ring of smooth functions on $J^r\pi$.

We denote by Ω^k_r the $\mathcal{F}_r$-module of k-forms on $J^r\pi$.

We denote by Ω^*_r the exterior algebra of forms on $J^r\pi$.

Definition 2.1. We say that a form $\alpha \in \Omega^k_r$ is a *contact k-form* if

$$(j_r s)^*\alpha = 0$$

for all sections s of π.

We denote by $\mathcal{C}^1\Omega^k_r$ the $\mathcal{F}_r$-module of contact k-forms on $J^r\pi$.

We denote by $\mathcal{C}^1\Omega^*_r$ the exterior algebra of contact forms on $J^r\pi$.

Note that if $k > n$ then every form is contact, *i.e.*, $\mathcal{C}^1\Omega^k_r = \Omega^k_r$.

It is obvious from the commutation of d and pull-back that $d\mathcal{C}^1\Omega^k_r \subset \mathcal{C}^1\Omega^{k+1}_r$. Moreover, it is obvious that $\mathcal{C}^1\Omega^*_r$ is an ideal of Ω^*_r. Hence, the following lemma holds.

Lemma 2.2. *The space $\mathcal{C}^1\Omega^*_r$ is a differential ideal of Ω^*_r.*

Unfortunately, the above ideal does not coincide with the ideal generated by 1-forms which annihilate the contact distribution (for this would contradict the non-integrability). More precisely, the following lemma can be easily proved (see, *e.g.*, [70]).

Lemma 2.3. *The space $\mathcal{C}^1\Omega^1_r$ is locally generated (on $\mathcal{F}_r$) by the 1-forms*

$$\omega^i_{\boldsymbol{\sigma}} \stackrel{def}{=} du^i_{\boldsymbol{\sigma}} - u^i_{\boldsymbol{\sigma}+\lambda}dx^\lambda, \quad 0 \le |\boldsymbol{\sigma}| \le r-1.$$

The above differential forms generate the annihilator of the contact distribution, which is an ideal of Ω^*_r. However, such an ideal is not differential, hence it does not coincide with $\mathcal{C}^1\Omega^*_r$. To realize it, the following formula can be easily proved

$$d\omega^i_{\boldsymbol{\sigma}} = -\omega^i_{\boldsymbol{\sigma}+\lambda} \wedge dx^\lambda, \tag{2}$$

from which it follows that, when $|\boldsymbol{\sigma}| = r-1$, then $d\omega^i_{\boldsymbol{\sigma}}$, which is a contact 2-form, cannot be expressed through the 1-forms of lemma 2.3 because $\omega^i_{\boldsymbol{\sigma}+\lambda}$ contains derivatives of order $r+1$.

The following theorem has been first conjectured in [35] ($\mathcal{C}^1\Omega$-hypothesis), then proved in [70, 71].

Theorem 2.4. *Let $k \geq 2$. The space $\mathcal{C}^1\Omega^k_r$ is locally generated (on $\mathcal{F}_r$) by the forms*

$$\omega^i_{\boldsymbol{\sigma}}, \quad d\omega^i_{\boldsymbol{\tau}}, \qquad 0 \leq |\boldsymbol{\sigma}| \leq r-1, \quad |\boldsymbol{\tau}| = r-1.$$

We can consider forms which are generated by p-th exterior powers of contact forms. More precisely, we have the following definition.

Definition 2.5. Let $p \geq 1$. We say that a form $\alpha \in \Omega^k_r$ is a *p-contact k-form* if it is generated by p-th exterior powers of contact forms.

We denote by $\mathcal{C}^p\Omega^k_r$ the $\mathcal{F}_r$-module of p-contact k-forms on $J^r\pi$.

We denote by $\mathcal{C}^p\Omega^*_r$ the exterior algebra of p-contact forms on $J^r\pi$.

Finally, we set $\mathcal{C}^0\Omega^*_r \overset{\text{def}}{=} \Omega^*_r$.

In other words, $\mathcal{C}^p\Omega^*_r$ is the p-th power of the ideal $\mathcal{C}^1\Omega^*_r$ in Ω^*_r. Of course, a 1-contact form is just a contact form. The following lemma is trivial.

Lemma 2.6. *Let $p \geq 0$. We have the inclusion*

$$\mathcal{C}^{p+1}\Omega^*_r \subset \mathcal{C}^p\Omega^*_r.$$

It follows that the space $\mathcal{C}^{p+1}\Omega^*_r$ is a differential ideal of $\mathcal{C}^p\Omega^*_r$, hence of Ω^*_r.

2.2 Horizontalization

Following the discussion in the Introduction, we would like to introduce a tool to extract from a form $\alpha \in \Omega^k_r$ the non-trivial part (to the purposes of calculus of variations). In other words, we would like to introduce a map whose kernel is precisely the set of contact forms. First of all, we observe that eq. (2) and Theorem 2.4 suggest that such a map can be constructed if we allow it to increase the jet order by 1. More precisely, it can be easily proved that the contact 1-forms $\omega^i_{\boldsymbol{\sigma}}$, with $0 \leq |\boldsymbol{\sigma}| \leq r-1$ generate a natural subbundle $C^*_r \subset T^*J^r\pi$ [122]. We have the following lemma (see [82, 98]).

Lemma 2.7. *We have the splitting*

$$J^{r+1}\pi \underset{J^r\pi}{\times} T^*J^r\pi = \left(J^{r+1}\pi \underset{M}{\times} T^*M\right) \underset{J^{r+1}\pi}{\oplus} C^*_{r+1}, \tag{3}$$

with projections

$$D^{r+1}\colon J^{r+1}\pi \to T^*M \underset{M}{\otimes} TJ^r\pi, \quad \omega^{r+1}\colon J^{r+1}\pi \to T^*J^r\pi \underset{J^r\pi}{\otimes} VJ^r\pi,$$

with coordinate expression

$$D^{r+1} = dx^\lambda \otimes D_\lambda = dx^\lambda \otimes \left(\frac{\partial}{\partial x^\lambda} + u^i_{\boldsymbol{\sigma}+\lambda}\frac{\partial}{\partial u^i_{\boldsymbol{\sigma}}}\right),$$
$$\omega^{r+1} = \omega^i_{\boldsymbol{\sigma}} \otimes \frac{\partial}{\partial u^i_{\boldsymbol{\sigma}}} = (du^i_{\boldsymbol{\sigma}} - u^i_{\boldsymbol{\sigma}+\lambda}dx^\lambda) \otimes \frac{\partial}{\partial u^i_{\boldsymbol{\sigma}}}.$$

Note that the above construction makes sense through the natural inclusions $VJ^r\pi \subset TJ^r\pi$ and $J^{r+1}\pi \times_M T^*M \subset T^*J^{r+1}\pi$, the latter being provided by $T^*\pi_r$.

From elementary multilinear algebra (see the Appendix) it turns out that we have the splitting

$$J^{r+1}\pi \times_{J^r\pi} \wedge^k T^*J^r\pi = \bigoplus_{p+q=k} \left(J^{r+1}\pi \underset{M}{\times} \wedge^q T^*M\right) \underset{J^{r+1}\pi}{\oplus} \wedge^p C^*_{r+1}.$$

Now, we observe that a form $\alpha \in \Omega^k_r$ fulfills

$$\pi^*_{r+1,r}(\alpha)\colon J^{r+1}\pi \to \wedge^k T^*J^r\pi \subset \wedge^k T^*J^{r+1}\pi,$$

where the inclusion is realized through the map $T^*\pi_{r+1,r}$. Hence, $\pi^*_{r+1,r}(\alpha)$ can be split into $k+1$ factors which, respectively, have 0 contact factors, 1 contact factor, ..., k contact factors. More precisely, let us denote by $\mathcal{H}^q_r$ the set of q-forms of the type

$$\alpha\colon J^r\pi \to \wedge^q T^*M.$$

We have the following proposition (for a proof, see [70, 122, 124]).

Proposition 2.8. *We have the natural decomposition*

$$\Omega^k_r \subset \bigoplus_{p+q=k} \mathcal{C}^p\Omega^p_{r+1} \wedge \mathcal{H}^q_{r+1},$$

with splitting projections

$$\mathrm{pr}^{p,q}\colon \Omega^k_r \to \mathcal{C}^p\Omega^p_{r+1}\wedge\mathcal{H}^q_{r+1}, \quad \mathrm{pr}^{p,q}(\alpha) = \left(\binom{p+q}{q} \odot^p i_{D^{r+1}} \odot \odot^q i_{\omega^{r+1}}\right)\circ\pi^*_{r+1,r},$$

where $i_{D^{r+1}}$, $i_{\omega^{r+1}}$ stand for contractions followed by a wedge product (see [98] and the Appendix).

Note that the above maps $\mathrm{pr}^{p,q}$ are not surjective. See [122] for more details.

Definition 2.9. We say the *horizontalization* to be the map

$$h^{p,q}\colon \mathcal{C}^p\Omega^{p+q}_r \to \mathcal{C}^p\Omega^p_{r+1}\wedge\mathcal{H}^q_{r+1}, \quad \alpha\mapsto \mathrm{pr}^{p,q}(\alpha).$$

Horizontalization is not surjective, unless $n=1$ [72]. We denote by

$$\overline{\Omega}^{p,q}_r \stackrel{\text{def}}{=} h^{p,q}(\mathcal{C}^p\Omega^{p+q}_r) \tag{4}$$

the image of the horizontalization; we say an element $\bar{\alpha}\in\overline{\Omega}^{0,q}_r$ to be a *horizontal form.*

Probably the first occurrence of horizontalization is in [69]. Of course, horizontalization is just the above projection on forms which have 0 contact factors. Note that, if $q>n$, then horizontalization is the zero map. In coordinates, if $0<q\leq n$, then

$$\alpha = \alpha^{\boldsymbol{\sigma}_1\cdots\boldsymbol{\sigma}_h}_{i_1\ \cdots i_h\ \lambda_{h+1}\cdots\lambda_q}\, du^{i_1}_{\boldsymbol{\sigma}_1}\wedge\cdots\wedge du^{i_h}_{\boldsymbol{\sigma}_h}\wedge dx^{\lambda_{h+1}}\wedge\cdots\wedge dx^{\lambda_q}$$

and

$$h^{0,q}(\alpha) = u^{i_1}_{\boldsymbol{\sigma}_1+\lambda_1}\cdots u^{i_h}_{\boldsymbol{\sigma}_h+\lambda_h}\alpha^{\boldsymbol{\sigma}_1\cdots\boldsymbol{\sigma}_h}_{i_1\ \cdots i_h\ \lambda_{h+1}\cdots\lambda_q}\, dx^{\lambda_1}\wedge\cdots\wedge dx^{\lambda_q}, \tag{5}$$

where $0\leq h\leq q$ (see [5, 70, 71, 122, 125]). Note that the above form is not the most general polynomial in $(r+1)$-st derivatives, even if $q=1$. For $q>1$ the skew-symmetrization in the indexes $\lambda_1,\ldots,\lambda_h$ yields a peculiar structure in the polynomial, in which the sums of all terms of the same degree are said to be *hyperjacobians* [90]. Note that the coordinate expressions of $h^{p,q}$ can be obtained in a similar way.

The technical importance of horizontalization is in the next two results.

Lemma 2.10. *Let $\alpha\in\Omega^q_r$, with $0\leq q\leq n$, and $s\colon M\to E$ be a section. Then*

$$(j_rs)^*(\alpha) = (j_{r+1}s)^*(h^{0,q}(\alpha))$$

Proposition 2.11. *Let $p\geq 0$. The kernel of $h^{p,q}$ coincides with $p+1$-contact q-forms,* i.e.*,*

$$\mathcal{C}^{p+1}\Omega^{p+q}_r = \ker h^{p,q}.$$

For a proof of both results, see, for example, [124, 125].

2.3 Horizontal and vertical differential

The above decomposition also affects the exterior differential. Namely, the pull-back of the differential can be split in two operators, one of which raises the contact degree by one, and the other raises the horizontal degree by one. More precisely, in view of proposition 2.8 and following [98], we introduce the maps

$$i_H \colon \Omega^k_r \to \Omega^k_{r+1}, \quad i_H = i_{D^{r+1}} \circ \pi^*_{r+1,r}, \tag{6a}$$

$$i_V \colon \Omega^k_r \to \Omega^k_{r+1}, \quad i_V = i_{\omega^{r+1}} \circ \pi^*_{r+1,r}. \tag{6b}$$

The maps i_H and i_V are two derivations along $\pi_{r+1,r}$ of degree 0. Together with the exterior differential d they yield two derivations along $\pi_{r+1,r}$ of degree 1, the *horizontal* and *vertical differential*

$$d_H \stackrel{\text{def}}{=} i_H \circ d - d \circ i_H : \Omega^k_r \to \Omega^k_{r+1},$$

$$d_V \stackrel{\text{def}}{=} i_V \circ d - d \circ i_V : \Omega^k_r \to \Omega^k_{r+1},$$

It can be proved (see [98]) that d_H and d_V fulfill the properties

$$d_H^2 = d_V^2 = 0, \qquad d_H \circ d_V + d_V \circ d_H = 0, \tag{7a}$$

$$d_H + d_V = (\pi^{r+1}_r)^* \circ d, \tag{7b}$$

$$(j_{r+1}s)^* \circ d_V = 0, \qquad d \circ (j_r s)^* = (j_{r+1}s)^* \circ d_H. \tag{7c}$$

The action of d_H and d_V on functions $f : J^rY \to \mathbb{R}$ and one–forms on J^rY uniquely characterizes d_H and d_V. We have the coordinate expressions

$$d_H f = D_\lambda f \, dx^\lambda = \left(\frac{\partial f}{\partial x^\lambda} + u^i_{\boldsymbol{\sigma}+\lambda} \frac{\partial f}{\partial u^i_{\boldsymbol{\sigma}}} \right) dx^\lambda, \tag{8a}$$

$$d_H dx^\lambda = 0, \qquad d_H du^i_{\boldsymbol{\sigma}} = -du^i_{\boldsymbol{\sigma}+\lambda} \wedge dx^\lambda, \qquad d_H \omega^i_{\boldsymbol{\sigma}} = -\omega^i_{\boldsymbol{\sigma}+\lambda} \wedge dx^\lambda, \tag{8b}$$

$$d_V f = \frac{\partial f}{\partial u^i_{\boldsymbol{\sigma}}} \, \omega^i_{\boldsymbol{\sigma}}, \tag{8c}$$

$$d_V dx^\lambda = 0\,, \qquad d_V du^i_{\boldsymbol{\sigma}} = d^i_{\boldsymbol{\sigma}+\lambda} \wedge dx^\lambda, \qquad d_V \omega^i_{\boldsymbol{\sigma}} = 0. \tag{8d}$$

We note that $d_H du^i_{\boldsymbol{\sigma}} = d_H \omega^i_{\boldsymbol{\sigma}}$.

2.4 Infinite order jets

From subsections 2.1, 2.2, 2.3, it is clear that there are fundamental operations in the geometry of jets which do not preserve the order. For this reason the first formulations of variational sequences were derived in infinite order frameworks (with a partial exception in [5]). At the level of forms, this amounts at defining spaces containing all forms on any arbitrary (but finite) order jet. At the level of vector fields, this is done by considering infinite sequences of tangent vectors which are related by the maps $T\pi_{r,s}$.

In what follows we will use the notions of projective (or inverse) system, projective (or inverse) limit, injective (or direct) system, injective (or direct) limit. Such notions can be found in any book of homological algebra (see, *e.g.*, [96]).

We start with the following definition. Consider the projective system

$$\cdots \xrightarrow{\pi_{r+2,r+1}} J^{r+1}\pi \xrightarrow{\pi_{r+1,r}} J^r\pi \xrightarrow{\pi_{r,r-1}} \cdots \xrightarrow{\pi_{1,0}} E \xrightarrow{\pi} M.$$

Definition 2.12. We define the *infinite order jet space* to be the projective limit

$$J^\infty\pi \stackrel{\text{def}}{=} \varprojlim J^r\pi.$$

Any element $\theta \in J^\infty\pi$ is a sequence of points $\{\theta_r\}_{r\geq 0}$, $\theta_r \in J^r\pi$, which are related by the projections of the system, $\pi_{r,s}(\theta_r) = \theta_s$, $r \geq s$. Hence, we have obvious projections

$$\pi_{\infty,r}\colon J^\infty\pi \to J^r\pi, \qquad \pi_\infty\colon J^\infty\pi \to M.$$

Any section $s\colon M \to E$ induces an element $j^\infty s(x) \in J^\infty\pi$, for $x \in M$, in an obvious way, and conversely, any element $\theta \in J^\infty\pi$ is of the form $\theta = j^\infty s(x)$, with $x = \pi_\infty(\theta)$, for a well-known result of analysis.

Several results can be proved on the infinite order jet: it has the structure of a bundle on E whose fibres are $\mathbb{R}^\infty$, the space of sequences of real numbers; local coordinates on $J^\infty\pi$ are $(x^\lambda, u^i_{\boldsymbol{\sigma}})$, where $0 \leq |\boldsymbol{\sigma}| < +\infty$; it is connected if E is connected, it is Hausdorff and second countable [98]; it is paracompact [102]. Unfortunately, $\mathbb{R}^\infty$ is a Fréchet space which cannot be made into a Banach space [98], hence several important parts of the theory of infinite dimensional Banach manifolds fail to be true. But, to the purposes of building a variational sequence, we need just the ability to deal with functions, tangent vectors and forms which are defined on any finite order jet space. This does not amount at defining all possible functions, tangent vectors, forms on $J^\infty\pi$, but only at considering their inductive or projective counterparts. This is a more or less implicit assumption in the literature; see [18] for an exposition which is close to the present one.

We begin with the projective structure of the tangent space. Namely, we have the following projective system

$$\cdots \xrightarrow{T\pi_{r+2,r+1}} TJ^{r+1}\pi \xrightarrow{T\pi_{r+1,r}} TJ^r\pi \xrightarrow{T\pi_{r,r-1}} \cdots \xrightarrow{T\pi_{1,0}} TE \xrightarrow{T\pi} TM.$$

We define the *tangent space* $TJ^\infty\pi$ to be the projective limit of the above projective system. Hence a tangent vector at $\theta \in J^\infty\pi$ is a sequence of vectors $\{\bar{X}, X_r\}_{k\geq 0}$ tangent to M and to $J^r\pi$ respectively such that $T\pi_r(X_r) = \bar{X}$, $T\pi_{r,s}(X_r) = X_s$ for all $r \geq s \geq 0$. Any tangent vector can be presented in coordinates as the formal sum

$$X = X^\lambda \frac{\partial}{\partial x^\lambda} + X^i_{\boldsymbol{\sigma}} \frac{\partial}{\partial u^i_{\boldsymbol{\sigma}}}, \qquad 0 \leq |\boldsymbol{\sigma}| < +\infty, \tag{9}$$

where X^λ, $X^i_{\boldsymbol{\sigma}} \in \mathbb{R}$. Of course we have obvious projections

$$T\pi_{\infty,r} \colon TJ^\infty\pi \to TJ^r\pi, \qquad T\pi_\infty \colon TJ^\infty\pi \to TM,$$

by which it is possible to define pull-back of forms on the infinite order jet. Moreover, we define the *vertical subbundle* $VJ^\infty\pi \subset TJ^\infty\pi$ as the subspace $VJ^\infty\pi \stackrel{\text{def}}{=} \ker T\pi_\infty$. It could also be introduced as a projective limit of finite-order vertical bundles. In coordinates, a vertical tangent vector can be expressed as in (9), with the condition $X^\lambda = 0$.

Analogously, we define the *cotangent space* $T^*J^\infty\pi$ to be the injective limit of the injective system $\cdots T^*J^r\pi \to T^*J^{r+1}\pi \cdots$. Hence a cotangent vector at $\theta \in J^\infty\pi$ is an equivalence class of the direct sum $\oplus_{r\in\mathbb{N}} T^*_{\boldsymbol{\theta}} J^r\pi$ under the following equivalence relation: for all α, $\beta \in \oplus_{r\in\mathbb{N}} T^*_{\boldsymbol{\theta}} J^r\pi$ we set $\alpha \sim \beta$ if and only if there exist r, $s \in \mathbb{N}$, $r < s$, such that $T^*\pi_{s,r}(\alpha) = \beta$. Moreover, we define the *horizontal subbundle* $\pi^*_\infty(T^*M) \subset T^*J^\infty\pi$.

Any tangent vector can be presented in coordinates as the formal sum

$$X = X^\lambda \frac{\partial}{\partial x^\lambda} + X^i_{\boldsymbol{\sigma}} \frac{\partial}{\partial u^i_{\boldsymbol{\sigma}}}, \qquad 0 \leq |\boldsymbol{\sigma}| < +\infty, \tag{10}$$

where X^λ, $X^i_{\boldsymbol{\sigma}} \in \mathbb{R}$. Any cotangent vector can be presented in coordinates as the finite sum

$$\alpha = \alpha_\lambda dx^\lambda + \alpha^{\boldsymbol{\sigma}}_i du^i_{\boldsymbol{\sigma}}, \qquad 0 \leq |\boldsymbol{\sigma}| \leq r, \tag{11}$$

for an $r \in \mathbb{N}$.

According with lemma 2.7, we have the following lemma (see [98]).

Lemma 2.13. *We have the splittings*

$$TJ^\infty\pi = C^\infty \underset{J^\infty\pi}{\oplus} VJ^\infty\pi, \tag{12}$$

$$T^*J^\infty\pi = \pi^*_\infty(T^*M) \underset{J^\infty\pi}{\oplus} C^*_\infty, \tag{13}$$

where

- *C^∞ is the projective limit of the projective system $\cdots C^{r+1} \to C^r \cdots$, where the projection is the restriction of $T\pi_{r+1,r}$ to C^{r+1};*

- *C^*_∞ is the injective limit of the injective system $\cdots C^*_r \to C^*_{r+1} \cdots$, where the injection is the restriction of $T^*\pi_{r+1,r}$ to C^*_r.*

The splitting projections are just the direct limit of the maps i_H and i_V of (6)*, that we indicate with the same symbol.*

Now, we could introduce functions and differential forms on $J^\infty\pi$ as functions on $J^\infty\pi$ or sections of exterior powers of $T^*J^\infty\pi$. But we prefer to insist with our 'injective limit' approach because it makes more clear the ideas exposed in the beginning of this section.

The composition with $\pi_{r+1,r}$ provides the injective system of rings $\cdots \subset \mathcal{F}_r \subset \mathcal{F}_{r+1} \subset \cdots$. We can regard the above system also as a *filtered algebra* [18]. Accordingly, pull-back via $\pi_{r+1,r}$ provides several injective (or direct) systems of modules over the above injective system of rings, namely $\cdots \subset \Omega^k_r \subset \Omega^k_{r+1} \subset \cdots$, $\cdots \subset \overline{\Omega}^{0,q}_r \subset \overline{\Omega}^{0,q}_{r+1} \subset \cdots$, $\cdots \subset \mathcal{C}^p\Omega^{p+q}_r \subset \mathcal{C}^p\Omega^{p+q}_{r+1} \subset \cdots$.

Let us introduce the injective (or direct) limits of the above injective systems

$$\mathcal{F} \stackrel{\text{def}}{=} \varinjlim \mathcal{F}_r, \qquad \Omega^k \stackrel{\text{def}}{=} \varinjlim \Omega^k_r, \qquad \overline{\Omega}^{0,q} \stackrel{\text{def}}{=} \varinjlim \overline{\Omega}^{0,q}_r, \qquad \mathcal{C}^p\Omega^{p+q} \stackrel{\text{def}}{=} \varinjlim \mathcal{C}^p\Omega^{p+q}_r.$$

Definition 2.14. We say:

- $f \in \mathcal{F}$ to be a *(smooth) function on $J^\infty\pi$*;

- $\alpha \in \Omega^k$ to be a *(differential) k-form on $J^\infty\pi$*;

- $\bar{\alpha} \in \overline{\Omega}^{0,q}$ to be a *horizontal q-form on $J^\infty\pi$*;

- $\gamma \in \mathcal{C}^p\Omega^k$ to be a *p-contact k-form on $J^\infty\pi$*.

From the definition of direct limit it follows that elements $f \in \mathcal{F}$ are equivalence classes of the direct sum $\oplus_{r\in\mathbb{N}}\mathcal{F}_r$ under the following equivalence relation: for all g, $h \in \oplus_{r\in\mathbb{N}}\mathcal{F}_r$ we set $g \sim h$ if and only if there exist r, $s \in \mathbb{N}$, $r < s$, such that $\pi^*_{s,r}h = g$. Of course, the same holds for the other spaces in the above definition, so that:

- $\mathcal{F}$ is made by all functions on a jet space $J^r\pi$ of arbitrary, but finite, order;

- Ω^k is made by all k-forms on a jet space $J^r\pi$ of arbitrary, but finite, order;

- $\overline{\Omega}^{0,q}$ is made by all horizontal q-forms on a jet space $J^r\pi$ of arbitrary, but finite, order; if $\alpha \in \overline{\Omega}^{0,q}$, then, locally,

$$\alpha = \alpha_{\lambda_1 \cdots \lambda_k}\, dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_k}, \qquad \alpha_{\lambda_1 \cdots \lambda_k} \in \mathcal{F};$$

  hence, if $\alpha \in \overline{\Omega}^{0,q}$ then $\alpha \colon J^r\pi \to \wedge^k T^*M$, for some $r \in \mathbb{N}$. For this reason, if we consider the inductive system $\cdots \subset \mathcal{H}^q_r \subset \mathcal{H}^q_{r+1} \subset \cdots$ and its injective limit $\mathcal{H}^q$, we have the equality $\mathcal{H}^q = \overline{\Omega}^{0,q}$, which *does not* holt at any finite order level;

- $\mathcal{C}^p\Omega^k$ is made by all p-contact k-forms on a jet space $J^r\pi$ of arbitrary, but finite, order; if $\alpha \in \mathcal{C}^p\Omega^k$, then, locally,

$$\alpha = \omega^{i_1}_{\boldsymbol{\sigma}_1} \wedge \cdots \wedge \omega^{i_p}_{\boldsymbol{\sigma}_p} \wedge \alpha^{\boldsymbol{\sigma}_1 \cdots \boldsymbol{\sigma}_p}_{i_1 \cdots i_p}, \qquad \alpha^{\boldsymbol{\sigma}_1 \cdots \boldsymbol{\sigma}_p}_{i_1 \cdots i_p} \in \Omega^{k-p},$$

  where the multiindexes $\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_p$ have arbitrary, but finite, length.

The differential d, the projections $\mathrm{pr}^{p,q}$ (hence also the horizontalization $h^{p,q}$) and the differentials d_H, d_V on finite order jets induce the maps

$$\begin{aligned} &d \colon \Omega^k \to \Omega^{k+1},\ \alpha \mapsto d\alpha, && \mathrm{pr}^{p,q} \colon \Omega^{p+q} \to \mathcal{C}^p\Omega^p \wedge \overline{\Omega}^{0,q},\ \alpha \mapsto \mathrm{pr}^{p,q}(\alpha), \\ &d_H \colon \Omega^k \to \Omega^{k+1},\ \alpha \mapsto d_H\alpha, && d_V \colon \Omega^k \to \Omega^{k+1},\ \alpha \mapsto d_V\alpha, \end{aligned}$$

for each $k \geq 0$, where, being $\alpha \in \Omega^k_r$ for some r, $d\alpha$ coincides with the differential of α on Ω^k_r, and analogously for $\mathrm{pr}^{p,q}$, d_H and d_V.

The proof of the following proposition follows easily from the definitions, the coordinate expressions (8), and proposition 2.8.

Proposition 2.15. *We have the natural splitting*

$$\Omega^k = \bigoplus_{p+q=k} \mathcal{C}^p\Omega^p \wedge \overline{\Omega}^{0,q};$$

with splitting projections

$$\mathrm{pr}^{p,q} \colon \Omega^{p+q} \to \mathcal{C}^p\Omega^p \wedge \overline{\Omega}^{0,q}, \quad \mathrm{pr}^{p,q}(\alpha) = \binom{p+q}{q} \odot^p i_V \odot \odot^q i_H.$$

Moreover, we have the following inclusions

$$d_H(\mathcal{C}^p\Omega^p \wedge \overline{\Omega}^{0,q}) \subset \mathcal{C}^p\Omega^p \wedge \overline{\Omega}^{0,q+1}, \qquad d_V(\mathcal{C}^p\Omega^p \wedge \overline{\Omega}^{0,q}) \subset \mathcal{C}^{p+1}\Omega^{p+1} \wedge \overline{\Omega}^{0,q}.$$

Remark 2.16. The above splitting represents one of the major differences between the finite order and the infinite order case. The simple structure of the splitting and the behaviour of d_H and d_V will allow us to give an easy definition of the variational sequence in the infinite order case.

Remark 2.17. The differentials d_H and d_V can also be defined through the above splitting. More precisely, it can be easily proved that

$$d(\mathcal{C}^p\Omega^p \wedge \overline{\Omega}^{0,q}) \subset \mathcal{C}^p\Omega^p \wedge \overline{\Omega}^{0,q+1} \oplus \mathcal{C}^{p+1}\Omega^{p+1} \wedge \overline{\Omega}^{0,q};$$

then d_H is the projection onto the first factor and d_V is the projection onto the second factor of the restriction of d to $\mathcal{C}^p\Omega^p \wedge \overline{\Omega}^{0,q}$ (see [4]).

Finally, a *vector field on* $J^\infty\pi$ is a filtered derivation of $\mathcal{F}$, *i.e.*, an $\mathbb{R}$-linear derivation $X\colon \mathcal{F} \to \mathcal{F}$ such that $X(\mathcal{F}_r) \subset \mathcal{F}_{r+s}$ for all r, and for $l \geq 0$ which depends on X. The number l is said to be the *filtration degree* of the field X. The set of all vector fields is a filtered Lie algebra over $\mathbb{R}$ with respect to commutator $[X, Y]$. Of course, any vector field X on $J^\infty\pi$ can be regarded as a section of $TJ^\infty\pi$ with coordinate expression

$$X = X^\lambda \frac{\partial}{\partial x^\lambda} + X^i_{\boldsymbol{\sigma}} \frac{\partial}{\partial u^i_{\boldsymbol{\sigma}}}, \qquad 0 \leq |\boldsymbol{\sigma}| < +\infty.$$

where $X^\lambda \in \mathcal{F}_s$ and $X^i_{\boldsymbol{\sigma}} \in \mathcal{F}_{r+s}$ [18].

Let X be a vector field on $J^\infty\pi$. Then X can be split according with (12) as

$$\begin{aligned} &X = X_H + X_V, && (14)\\ &X_H = X^\lambda D_\lambda, \qquad X_V = (X^i_{\boldsymbol{\sigma}} - u^i_{\boldsymbol{\sigma}+\lambda} X^\lambda) \frac{\partial}{\partial u^i_{\boldsymbol{\sigma}}} \quad 0 \leq |\boldsymbol{\sigma}| < +\infty. \end{aligned}$$

We observe that any vector field $X\colon E \to TE$ which projects onto a vector field $X\colon M \to TM$ can be prolonged to a vector field X^∞ with filtration degree 0. We have

$$X^\infty = X^\lambda D_\lambda + D_{\boldsymbol{\sigma}}(X^i - u^i_\lambda X^\lambda) \frac{\partial}{\partial u^i_{\boldsymbol{\sigma}}}, \quad 0 \leq |\boldsymbol{\sigma}| < +\infty, \tag{15}$$

where $X^\lambda \in C^\infty(M)$. The vector field X^∞_V is said to be the *evolutionary vector field* with *generating function* X (see, *e.g.*, [18, 91]).

We can consider more general evolutionary vector fields. Namely, it can be proved (see, *e.g.*, [18, 91]) that a vector field X on $J^\infty\pi$ is a symmetry of C^∞ if and only if it its vertical part is of the form $X_V = E_\varphi$, where $\varphi\colon J^r\pi \to V\pi$ and

$$E_\varphi\colon J^\infty\pi \to VJ^\infty\pi, \quad E_\varphi = D_{\boldsymbol{\sigma}}\varphi^i \frac{\partial}{\partial u^i_{\boldsymbol{\sigma}}}. \tag{16}$$

We say E_φ to be an evolutionary vector field with generating function φ; of course, the filtration degree of E_φ is the order r of the jet space on which

φ is defined. It can be proved [18, 91] that evolutionary vector fields are uniquely determined by their generating functions. We denote the $\mathcal{F}_r$-module of generating functions on $J^r\pi$ with $\varkappa_r$. Composing with projections $\pi_{r+1,r}$ yields the chain of inclusions $\cdots \subset \varkappa_r \subset \varkappa_{r+1} \subset \cdots$, hence the direct limit $\varkappa$. This module plays an important role in section 4.

3 Variational bicomplex and variational sequence

Variational sequences has been introduced basically in two ways.

The first way is through the properties of d_H, d_V on infinite order jets [108, 109, 102, 110]. Another way to describe this approach is to consider the splitting (12) as a connection on $J^\infty\pi$ which has zero curvature.

The second way is through a spectral sequence [29, 30, 115, 116, 118]; this approach will be described in section 4.

Partial exceptions to this classification are [5], where the approach is (at least partially) on finite order jets, and [17], where the approach is based on the properties of the interior Euler operator (see subsection 3.1). In this section we describe the approach of [108, 109, 102, 110] in a modern language which is close to that of [4, 98].

For all $p \geq 0$ we introduce the following notation:

$$E_0^{0,q} \stackrel{\text{def}}{=} \overline{\Omega}^{0,q}, \qquad E_0^{p,q} \stackrel{\text{def}}{=} \mathcal{C}^p\Omega^p \wedge \overline{\Omega}^{0,q}, \tag{17}$$

$$E_1^{p,n} \stackrel{\text{def}}{=} E_0^{p,n} / d_H(E_0^{p,n-1}) = \mathcal{C}^p\Omega^p \wedge \overline{\Omega}^{0,n} / d_h(\mathcal{C}^p\Omega^p \wedge \overline{\Omega}^{0,n-1}). \tag{18}$$

The integers p, q are called, respectively, the *contact* and the *horizontal degree.*

We also denote by $\Omega^k(M)$ the space of k-forms on M.

We define the map

$$e_1 \colon E_1^{p,n} \to E_1^{p+1,n}, \quad e_1([\alpha]) = [d_V\alpha]. \tag{19}$$

The above map is well-defined because $d_V \circ d_H = -d_H \circ d_V$.

In view of the properties (7a) of d_H and d_V the following diagram commutes

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & E_0^{0,0} \arrow[r, "d_V"] \arrow[d, "d_H"] & E_0^{1,0} \arrow[r, dotted, dash] \arrow[d, "-d_H"] & E_0^{p,0} \arrow[r, "d_V"] \arrow[d, "(-1)^p d_H"] & E_0^{p+1,0} \arrow[r, dotted, dash] \arrow[d, "(-1)^{p+1} d_H"] & {} \\
0 \arrow[r] & E_0^{0,1} \arrow[r, "d_V"] \arrow[d, dotted, dash] & E_0^{1,1} \arrow[r, dotted, dash] \arrow[d, dotted, dash] & E_0^{p,1} \arrow[r, "d_V"] \arrow[d, dotted, dash] & E_0^{p+1,1} \arrow[r, dotted, dash] \arrow[d, dotted, dash] & {} \\
0 \arrow[r] & E_0^{0,n-1} \arrow[r, "d_V"] \arrow[d, "d_H"] & E_0^{1,n-1} \arrow[r, dotted, dash] \arrow[d, "-d_H"] & E_0^{p,n-1} \arrow[r, "d_V"] \arrow[d, "(-1)^p d_H"] & E_0^{p+1,n-1} \arrow[r, dotted, dash] \arrow[d, "(-1)^{p+1} d_H"] & {} \\
0 \arrow[r] & E_0^{0,n} \arrow[r, "d_V"] \arrow[d] & E_0^{1,n} \arrow[r, dotted, dash] \arrow[d] & E_0^{p,n} \arrow[r, "d_V"] \arrow[d] & E_0^{p+1,n} \arrow[r, dotted, dash] \arrow[d] & {} \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}
\tag{20}
$$

and rows and columns are complexes (in the sense that the kernel of a map contains the image of the previous). According with standard terminology from homological algebra, the above diagram (20) is a *double complex*, or a *bicomplex* [20]. The diagram (20) can be *augmented* (again, a standard procedure from homological algebra) by the natural inclusion of de Rham complex of M on the left edge and the natural quotient projection on the complex

$$
0 \longrightarrow E_1^{0,n} \xrightarrow{e_1} E_1^{1,n} \cdots\cdots E_1^{p,n} \xrightarrow{e_1} E_1^{p+1,n} \cdots\cdots
$$

on the bottom edge. The resulting bicomplex is

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & & & & \\
 & \mathbb{R} \arrow[d] & \mathbb{R} \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^0(M) \arrow[r, "\pi_\infty^*"] \arrow[d, "d"] & E_0^{0,0} \arrow[r, "d_V"] \arrow[d, "d_H"] & E_0^{1,0} \arrow[d, "-d_H"] & \cdots & E_0^{p,0} \arrow[r, "d_V"] \arrow[d, "(-1)^p d_H"] & E_0^{p+1,0} \arrow[d, "(-1)^{p+1} d_H"] & \cdots \\
0 \arrow[r] & \Omega^1(M) \arrow[r, "\pi_\infty^*"] & E_0^{0,1} \arrow[r, "d_V"] & E_0^{1,1} & \cdots & E_0^{p,1} \arrow[r, "d_V"] & E_0^{p+1,1} & \cdots \\
 & \vdots & \vdots & \vdots & & \vdots & \vdots & \\
0 \arrow[r] & \Omega^{n-1}(M) \arrow[r, "\pi_\infty^*"] \arrow[d, "d"] & E_0^{0,n-1} \arrow[r, "d_V"] \arrow[d, "d_H"] & E_0^{1,n-1} \arrow[d, "-d_H"] & \cdots & E_0^{p,n-1} \arrow[r, "d_V"] \arrow[d, "(-1)^p d_H"] & E_0^{p+1,n-1} \arrow[d, "(-1)^{p+1} d_H"] & \cdots \\
0 \arrow[r] & \Omega^n(M) \arrow[r, "\pi_\infty^*"] \arrow[d] & E_0^{0,n} \arrow[r, "d_V"] \arrow[d] & E_0^{1,n} \arrow[d] & \cdots & E_0^{p,n} \arrow[r, "d_V"] \arrow[d] & E_0^{p+1,n} \arrow[d] & \cdots \\
 & 0 \arrow[r] & E_1^{0,n} \arrow[r, "e_1"] \arrow[d] & E_1^{1,n} \arrow[d] & \cdots & E_1^{p,n} \arrow[r, "e_1"] \arrow[d] & E_1^{p+1,n} \arrow[d] & \cdots \\
 & & 0 & 0 & & 0 & 0 &
\end{tikzcd} \tag{21}$$

Definition 3.1. We say the *variational bicomplex* associated with the fibred manifold $\pi\colon E \to M$ to be the bicomplex (21).

The variational sequence can be extracted from the variational bicomplex.

Definition 3.2. The following complex

$$\begin{aligned} 0 \longrightarrow \mathbb{R} \longrightarrow E_0^{0,0} \xrightarrow{d_H} E_0^{0,1} \cdots E_0^{0,n-1} \xrightarrow{d_H} E_0^{0,n} \xrightarrow{\mathcal{E}} \\ \xrightarrow{\mathcal{E}} E_n^{1,n} \xrightarrow{e_1} E_1^{2,n} \cdots E_1^{p,n} \xrightarrow{e_1} E_1^{p+1,n} \cdots \end{aligned} \tag{22}$$

where the map $\mathcal{E}$ is just the composition of the quotient projection $E_0^{0,n} \to E_1^{0,n}$ with the differential $e_1\colon E_1^{0,n} \to E_1^{1,n}$, is said to be the *variational sequence*[3].

[3]Some authors use the term *Euler–Lagrange complex* instead, see [4]

The motivation for the above definition is immediate after the analysis of the quotient spaces $E_1^{p,q}$ that we are going to perform in next subsection.

The second column of the variational bicomplex has a special importance and will be studied later on.

Definition 3.3. We say the following sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow E_0^{0,0} \xrightarrow{d_H} E_0^{0,1} \cdots\cdots E_0^{0,n-1} \xrightarrow{d_H} E_0^{0,n} \longrightarrow 0 \tag{23}$$

to be the *horizontal de Rham sequence.*

3.1 Representation of the variational sequence by forms

The problem of representing the elements of the quotients $E_1^{p,n}$ for $p > 1$ has been independently solved by many authors [109, 110, 115, 116, 79, 17]. We recognize two different approaches to the problem: with differential forms [109, 110, 79, 17] and with differential operators [115, 116]. In this section we follow the first approach. The interpretation of the variational sequence (22) in terms of objects of the calculus of variations will follow at once.

Following [109, 110, 4], let us introduce the map

$$I\colon E_0^{p,n} \to E_0^{p,n}, \quad I(\alpha) = \frac{1}{p}\omega^i \wedge (-1)^{|\boldsymbol{\sigma}|} D_{\boldsymbol{\sigma}} \left(i_{\partial/\partial u^i_{\boldsymbol{\sigma}}} \alpha \right) \tag{24}$$

where $D_{\boldsymbol{\sigma}}$ stands for the iterated Lie derivative $(L_{D_1})^{\sigma_1} \cdots (L_{D_n})^{\sigma_n}$.

Definition 3.4. We say the map I to be the *interior Euler operator*[4].

Note that I is denoted by τ in [109, 110] and by D^* in [17]. For a proof of the following theorem, see [4, 67, 110].

Theorem 3.5. *The following properties of I holds*

- *I is a natural map,* i.e., *$L_{X^\infty}(I(\alpha)) = I(L_{X^\infty}(\alpha))$, hence I is a global map;*
- *if $\alpha \in E_0^{p,n}$ then there exists a unique form $\beta \in E_0^{n,p}$, which is of the type $\beta = d_H\gamma$ with $\gamma \in E_0^{p,n-1}$, such that*

$$\alpha = I(\alpha) + \beta. \tag{25}$$

[4] This name is due to I. Anderson.

Remark 3.6. The above form γ is not uniquely defined, in general. For $p = 1$, if the order of α is 1 it is easily proved that γ is uniquely defined; if the order of α is 2 then there exists a unique γ fulfilling a certain intrinsic property; if the order is 3 it is proved in [61] that no natural γ of the above type exists. In [37, 41, 61] it is proved that suitable linear connections on M and on the fibres of $\pi\colon E \to M$ can be used to determine a unique γ. See [2, 4] for the case $p > 1$.

It follows from the above theorem that I is a global map, and if $\gamma \in E_1^{p,n-1}$ then $I(d_H(\gamma)) = 0$, so that $I^2 = I$. For this reason I induces an isomorphism (denoted by the same letter)

$$I\colon E_1^{p,n} \to \mathcal{V}^p, \quad [\alpha] \mapsto I(\alpha),$$

where $\mathcal{V}^p \subset E_0^{p,n}$ is a suitable subspace. The map I also allows us to represent the differentials $\mathcal{E}$, e_1 through forms: $I(\mathcal{E}(\lambda)) = I([d_V \lambda])$, and $I(e_1([\alpha])) = I([d_V \alpha])$.

Definition 3.7. We say the elements of $\mathcal{V}^p$ to be the p-th degree *variational (or functional, as in [4]) forms.*

Let us see the coordinate expression of I in the most meaningful cases. We set $\nu \overset{\text{def}}{=} dx^1 \wedge \cdots \wedge dx^n$.

Case $p = 1$: let $[\alpha] \in E_1^{1,n}$. Then $\alpha = \alpha_i^{\boldsymbol{\sigma}} \omega^i_{\boldsymbol{\sigma}} \wedge \nu$ and

$$I([\alpha]) = (-1)^{|\boldsymbol{\sigma}|} D_{\boldsymbol{\sigma}} \alpha_i^{\boldsymbol{\sigma}}\, \omega^i \wedge \nu.$$

Hence, if $\lambda \in E_0^{0,n}$, then $\lambda = L\nu$, $\mathcal{E}(\lambda) = [\partial L / \partial u^i_{\boldsymbol{\sigma}} \omega^i_{\boldsymbol{\sigma}} \wedge \nu]$ and

$$I(\mathcal{E}(\lambda)) = (-1)^{|\boldsymbol{\sigma}|} D_{\boldsymbol{\sigma}} \frac{\partial L}{\partial u^i_{\boldsymbol{\sigma}}}\, \omega^i \wedge \nu, \tag{26}$$

which is just the expression of the *Euler–Lagrange form* corresponding to the *Lagrangian form* λ. It can be proved that the Euler–Lagrange form is the only natural operator in a broad class of differential operators [62, 63]. It is not difficult to prove the following result (see [98, 102]).

Proposition 3.8. *The space $\mathcal{V}^1$ is equal to the injective limit of the system $\cdots \mathcal{V}^1_r \subset \mathcal{V}^1_{r+1} \cdots$, where $\mathcal{V}^1_r$ is the space of sections of the bundle*

$$(\pi^*_{r,1} C^*_1) \wedge (\pi^*_r \wedge^n T^* M)\,.$$

Following [102], the elements of $\mathcal{V}^1$ are called *source forms.* A source form $\eta \in \mathcal{V}^1$ has the coordinate expression

$$\eta = \eta_i\, \omega^i \wedge \nu, \qquad \eta_i \in \mathcal{F},\ i = 1, \dots, m.$$

Case $p = 2$**:** let $[\alpha] \in E_1^{1,n}$. Then $\alpha = \alpha_{i\ j}^{\boldsymbol{\sigma}\boldsymbol{\tau}} \omega^i_{\boldsymbol{\sigma}} \wedge \omega^j_{\boldsymbol{\tau}} \wedge \nu$ (with $\alpha_{i\ j}^{\boldsymbol{\sigma}\boldsymbol{\tau}} = -\alpha_{j\ i}^{\boldsymbol{\tau}\boldsymbol{\sigma}}$) and, if α is a form on the r-th order jet, then

$$\begin{aligned} I([\alpha]) &= \frac{1}{2}\omega^i \wedge (-1)^{|\boldsymbol{\sigma}|} D_{\boldsymbol{\sigma}}(\alpha_{i\ j}^{\boldsymbol{\sigma}\boldsymbol{\tau}}\, \omega^j_{\boldsymbol{\tau}}) \wedge \nu \\ &= \frac{1}{2} \sum_{\substack{\boldsymbol{\mu}+\boldsymbol{\tau}=\boldsymbol{\rho} \\ 0\leq|\boldsymbol{\rho}|\leq 2r}} (-1)^{|\boldsymbol{\xi}+\boldsymbol{\mu}|} \frac{(\boldsymbol{\xi}+\boldsymbol{\mu})!}{\boldsymbol{\xi}!\boldsymbol{\mu}!} D_{\boldsymbol{\xi}} \alpha_{i\ \ j}^{\boldsymbol{\xi}+\boldsymbol{\mu}\boldsymbol{\tau}} \omega^i \wedge \omega^j_{\boldsymbol{\rho}} \wedge \nu. \end{aligned} \tag{27}$$

If $\eta \in \mathcal{V}^1$, then η represents the element $[\eta] \in E_1^{p,q}$. Hence, if η is a form on the r-th order jet, then

$$\begin{aligned} I(e_1[\eta]) =& I(d_V\eta) \\ =& I\left(\frac{\partial \eta_k}{\partial u^h_{\boldsymbol{\sigma}}} \omega^h_{\boldsymbol{\sigma}} \wedge \omega^k \wedge \nu\right) \\ =& \frac{1}{2}\omega^i \wedge \left(\frac{\partial \eta_j}{\partial u^i} - \frac{\partial \eta_i}{\partial u^j}\right) \omega^j \wedge \nu \\ &+ \frac{1}{2}\omega^i \wedge \sum_{\substack{0\leq\boldsymbol{\rho}\leq 2r \\ |\boldsymbol{\mu}+\boldsymbol{\rho}|\geq 1}} (-1)^{|\boldsymbol{\mu}+\boldsymbol{\rho}|} \frac{(\boldsymbol{\mu}+\boldsymbol{\rho})!}{\boldsymbol{\mu}!\boldsymbol{\rho}!} D_{\boldsymbol{\mu}} \frac{\partial \eta_j}{\partial u^i_{\boldsymbol{\mu}+\boldsymbol{\rho}}} \omega^j_{\boldsymbol{\rho}} \wedge \nu \end{aligned} \tag{28}$$

The above form is the well-known *Helmholtz form* corresponding to the source form η. The above coordinate expression dates back to [17], even if the local expression of the Helmholtz conditions $I(d_V\eta) = 0$ of local variationality of η were known much before, even in the general case of arbitrary values of r and n.

The Helmholtz conditions may be also expressed by the Helmholtz tensor [64]. It has the same components of the Helmholtz form without skew-symmetrization with respect to the pair of indexes (i, j). It has been proved that the Helmholtz tensor is the only natural operator in a broad class [64, 87]. Note that the Helmholtz form is also connected with the second variation of functionals [39].

Note that if $p \geq 2$ then the spaces $\mathcal{V}^p$ cannot be characterized as spaces of sections of a vector bundle, like $\mathcal{V}^1$. This can be realized by the fact that $\mathcal{V}^p$ fail to be modules over $\mathcal{F}$. We will see in section 4 how to characterize the elements of $\mathcal{V}^p$.

The forms in $\mathcal{V}^p$ may also be interpreted as functionals. The case $p = 1$ was clear in the papers [40, 45, 69]; the case $p > 1$ was dealt with first in [17] (see also [4]). This provides a relationship between 'standard' calculus of variations and the theory of variational sequences.

Definition 3.9. Let $\alpha \in E_0^{p,n}$. Then we define the family of functionals $\mathcal{A}(\alpha)$

$$\mathcal{A}(\alpha)(X_1, \dots, X_p)(s)_U \stackrel{\text{def}}{=} z \int_U (j^\infty s)^* \alpha(X_1^\infty, \dots, X_p^\infty),$$

depending on an oriented open set with compact closure and oriented regular boundary $U \subset M$, on p vertical vector fields $\{X_i \colon E \to VE\}_{1 \le i \le p}$ which vanish on $\pi^{-1}(\partial U)$, and on a section s of π.

The vector fields $X_1, \dots, X_p$ are called *variation fields*. We denote by

$$\mathcal{F}^p \stackrel{\text{def}}{=} \{\mathcal{A}(\alpha) \mid \alpha \in E_0^{p,n}\}$$

the space of functionals.

The following proposition is proved in [4, 17].

Proposition 3.10. *Let α, $\alpha' \in E_0^{p,n}$. Then $\mathcal{A}(\alpha) = \mathcal{A}(\alpha')$ if and only if $[\alpha] = [\alpha'] \in E_1^{p,n}$, or, equivalently, if and only if $\alpha' = \alpha + d_H\beta$, with $\beta \in E_0^{p,n-1}$. Hence $\mathcal{V}^p \simeq \mathcal{F}^p$.*

Of course, in the case $p = 1$ we recover the standard integral of a source form evaluated on a variation field.

3.2 Local properties of the variational bicomplex

In this section we show that the variational bicomplex is locally exact. More precisely, recall that an exact sequence is a complex where the kernel of each map is equal to the image of the previous one. Then, we prove that for all $p \in E$ there exists an open neighbourhood $U \subset E$ of p such that the variational sequence on the fibred manifold $\pi|_U \colon U \to \pi(U)$ is an exact sequence.

We begin by proving an exactness result for the rows of the variational bicomplex.

Theorem 3.11. *Let $q \geq 0$. Then for all $p \in E$ there exists an open neighbourhood $U \subset E$ of p such that the rows*

$$0 \longrightarrow \Omega^q(M) \xrightarrow{\pi_\infty^*} E_0^{0,q} \xrightarrow{d_V} E_0^{1,q} \cdots\cdots E_0^{p,q} \xrightarrow{d_V} E_0^{p+1,q} \cdots\cdots$$

of the variational bicomplex associated with the fibred manifold $\pi|_U \colon U \to \pi(U)$ are exact.

The above theorem was proved in [109] for the case of jets of n-velocities (see, *e.g.*, [63] for a definition) and [110] for the case of jets of fibred manifolds (see also [4, 98] for a detailed exposition). The proof is just a 'vertical' version of the Poincaré lemma. In [115, 116] an alternative proof was proposed, see section 4.

A more complex homotopy operator is constructed in order to prove next theorem. Several proofs of the following result has been provided: [109, 110] (see also [4, 98] for a detailed exposition) and [102] with two different homotopy operators, [115, 116] with Spencer sequences, [106] with Koszul complexes (indeed, in [115, 116] the authors proved the *global exactness*, see also [118, 18] and section 4 for a detailed exposition).

Theorem 3.12. *Let $p \geq 1$. Then for all $p \in E$ there exists an open neighbourhood $U \subset E$ of p such that the columns*

$$0 \longrightarrow E_0^{p,0} \xrightarrow{(-1)^p d_H} E_0^{p,1} \cdots\cdots E_0^{p,n-1} \xrightarrow{(-1)^p d_H} E_0^{p,n} \longrightarrow 0$$

of the variational bicomplex associated with the fibred manifold $\pi|_U : U \to \pi(U)$ are exact for horizontal degrees $0 \leq q \leq n-1$.

Note that in the case $p = 1$ the global exactness was also established in [61] (see also references therein) by direct computation of the potential of d_H-closed forms using an auxiliary symmetric linear connection on M.

As relatively trivial consequences of the above theorems we have the following corollary, obtained via 'diagram chasing' techniques [109, 110, 102] (see also [4, 98] for a detailed exposition), or via spectral sequences [115, 116] (see also [118, 18] and section 4 for a detailed exposition).

Corollary 3.13. *For all $p \in E$ there exists an open neighbourhood $U \subset E$ of p such that in the variational bicomplex associated with the fibred manifold $\pi|_U : U \to \pi(U)$ the following complexes are exact:*

- *the horizontal de Rham sequence* (23)*;*
- *the bottom row of the variational sequence*

$$0 \longrightarrow E_1^{0,n} \xrightarrow{e_1} E_1^{1,n} \cdots\cdots E_1^{p,n} \xrightarrow{e_1} E_1^{p+1,n} \cdots\cdots \quad . \tag{29}$$

It follows that the variational sequence associated with the fibred manifold $\pi|_U : U \to \pi(U)$ is exact.

Note that also in [79] a variational sequence is constructed and the local exactness at the vertices of degree n and $n+1$ is proved.

3.3 Global properties of the variational bicomplex

In this section we collect results about the cohomology of rows and columns of the variational bicomplex on the given (but arbitrary) fibred manifold $\pi\colon E \to M$. We recall that the cohomology of a complex is the sequence of the quotients of the kernel of a map with the image of the previous one. The cohomology of the columns is the most studied because it allows to compute the cohomology of the variational sequence.

Theorem 3.14. *Let $p \geq 1$. Then the columns*

$$0 \longrightarrow E_0^{p,0} \xrightarrow{(-1)^p d_H} E_0^{p,1} \cdots\cdots E_0^{p,n-1} \xrightarrow{(-1)^p d_H} E_0^{p,1} \longrightarrow E_1^{p,n} \longrightarrow 0$$

are exact (i.e.*, the above sequence have zero cohomology) for horizontal degrees $0 \leq q \leq n-1$.*

The above theorem has been proved in several ways. The first proofs appeared in [115, 116] (using Spencer sequences; see also the longer paper [118]), in [102] (using a sheaf-theoretical approach) in [106] (using an isomorphism with the polynomial Koszul complex; see also the more modern texts [18, 66, 114, 121]) and in [4] (using local exactness and a Mayer–Vietoris argument). Note that the approach of [102] implies passing from modules of sections $E_0^{p,q}$ to the corresponding sheaves of germs of local sections. Those sheaves consists of sections which are defined on finite order jet spaces only locally (see [43, 123]). The following corollary holds.

Corollary 3.15. *The cohomology of the variational sequence is (not naturally) isomorphic to the de Rham cohomology of E.*

Note that the above corollary implies that the cohomology of the horizontal de Rham sequence is isomorphic to the de Rham cohomology of E for horizontal degrees $0 \leq q \leq n-1$. Such a cohomology is also called *characteristic cohomology* in the framework of exterior differential systems [22, 23].

The above corollary can be proved using spectral sequences [115, 116] (but see the more modern texts [18, 66, 114, 121]), sheaf-theoretical arguments [102] or just basic diagram chasing [4]. Note that all proofs show first that the cohomology of the variational sequence is isomorphic to the cohomology of the complex (Ω^*, d), which is, by definition, the de Rham cohomology $H^*(J^\infty\pi)$ of $J^\infty\pi$. Then it is quickly seen that $H^*(J^\infty\pi)$ is isomorphic to $H^*(E)$, just by the fact that, in this case, the cohomology functor commutes with direct limits.

The cohomology of the rows of the variational bicomplex is much less studied. We have the following results [4].

Theorem 3.16. *The cohomology of the rows*

$$0 \longrightarrow \Omega^q(M) \xrightarrow{\pi_\infty^*} E_0^{0,q} \xrightarrow{d_V} E_0^{1,q} \cdots\cdots E_0^{p,q} \xrightarrow{d_V} E_0^{p+1,q} \cdots\cdots$$

vanish for vertical degrees $p > m$.

Some restrictions on the topology of E have to be asked in order to compute the cohomology of vertical rows.

Theorem 3.17. *Let π be a bundle with typical fibre F. Suppose that F admit a finite covering $\{U_i\}_{0\leq i\leq k}$ such that each U_i and each non-empty intersection $U_{i_1} \cap \cdots \cap U_{i_l}$ is diffeomorphic to $\mathbb{R}^n$ for any l (finite* good cover*, see [20]). Suppose that for each p there are a finite number $\{\beta_i\}_{1\leq i\leq d}$ of p-forms on E whose restriction to the fibres of E is a basis for the cohomology of the fibres. Then*

$$H(E_0^{p,q}, d_V) \simeq H^p(F) \otimes H^q(M).$$

More precisely, the forms $\alpha_i \overset{def}{=} \pi_{\infty,0}^*(\beta_i) \in E_0^{p,0}$ are d_V-closed, and if $\alpha \in E_0^{p,q}$ is d_V-closed, then there are forms $\{\xi_i\}_{1\leq i\leq d}$ in $\Omega^q(M)$ and a form $\eta \in E_0^{p-1,q}$ such that

$$\alpha = \sum_{i=1}^{d} \xi_i \wedge \alpha_i + d_V\eta.$$

The forms $\{\xi_i\}_{1\leq i\leq d}$ are unique in the sense that α is d_V-exact if and only if $\{\xi_i\}_{1\leq i\leq d}$ vanish.

The above theorem is clearly inspired by the Leray–Hirsch theorem [20], but its hypotheses are weaker because the forms β_i are not assumed to be closed on E.

4 $\mathcal{C}$-spectral sequence and variational sequence

In this section we derive the variational sequence as a by-product of a spectral sequence, the $\mathcal{C}$-spectral sequence. To the author's knowledge the first formulations of the $\mathcal{C}$-spectral sequence (on infinite order jets) were done in [29] and [116], independently. But the computation of all terms of the $\mathcal{C}$-spectral sequence was done in [116] (using results from [115]), in the more general setting of differential equations (see also the longer paper [118]). Note that the variational sequence was already formulated in [115], without using the $\mathcal{C}$-spectral sequence. See the notes in section 7 for more details.

The $\mathcal{C}$-spectral sequence allows us not only to recover the variational bicomplex as was formulated in the previous section, but also to formulate a variational sequence on infinite order jets of submanifolds and on infinite prolongations of (ordinary or partial) differential equations (which, we recall, are submanifolds of a jet space of a certain finite order). In this section we will recall the main results on the $\mathcal{C}$-spectral sequence on the fibred manifold π. We will follow the most recent presentation of the subject [18, 66].

We will use the language of differential operators, as in [115, 116]. There are a number of reasons for doing that. First of all this language is used by a part of the scientists that work in this field. Then, it yields the same construction as the interior Euler operator using the adjoint of a differential operator. Moreover, differential operators and the operations on them constitute a calculus which is complementary to that of differential forms and is of independent interest with respect to variational sequences. An important domain of application of this calculus is, for example, the Hamiltonian formalism for evolution equations [56, 66].

4.1 The $\mathcal{C}$-spectral sequence and its 0-th term

Here we introduce the $\mathcal{C}$-spectral sequence and compute its first term.

We begin by recalling the basic facts on spectral sequences, but we suggest the interested reader to consult a book on algebraic topology (like [20, 81]; see also [66]).

We recall that a *filtered module* is a module P endowed with a chain[5] of submodules

$$P \stackrel{\text{def}}{=} F^0P \supset F^1P \supset F^2P \supset \cdots \supset F^pP \supset \cdots$$

A filtered module yields the *associated graded module* $S_0^*(P)$, where

$$S_0^p(P) \stackrel{\text{def}}{=} F^pP \big/ F^{p+1}P.$$

A (graded) *filtered complex* is a (graded) filtered module P endowed with a differential d of degree 1 which preserves the filtration, *i.e.*, $d(F^pP) \subset F^pP$. With every (graded) filtered complex it is associated a filtration of its cohomology $H^*(P)$ as follows:

$$H^*(P) = F^0H^*(P) \supset F^1H^*(P) \supset F^2H^*(P) \supset \cdots \supset F^pH^*(P) \supset \cdots \quad (30)$$

where $F^pH^*(P)$ is the image of the cohomological map $H^*(F^pP) \to H^*(P)$ induced by the inclusion $F^pP \subset P$. In general (30) is a filtration without a natural differential.

[5] We will only use decreasing filtrations.

Any filtered complex yields a *spectral sequence*. A spectral sequence is a sequence of differential Abelian groups (S_n, s_n) where the cohomology of each term is equal to the next term: $H(S_n, s_n) = S_{n+1}$.

A spectral sequence is said to *converge* if there exists $k \in \mathbb{N}$ such that for every $k' \in \mathbb{N}$, $k' > k$, we have $S_k = S_{k'}$. In this case we set $S_\infty \stackrel{\text{def}}{=} S_k$. For spectral sequences associated with filtered complexes the notion of convergence is more specific. Namely, a spectral sequence associated with a filtered complex is said to *converge* if it exists a graded filtered module Q such that $S_\infty = S_0^*(Q)$. It can be proved [20, 81, 66] that if a spectral sequence associated with a filtered complex P lays in the *first quadrant* (*i.e.*, $S_r^{p,q} = 0$ whenever $p < 0$ or $q < 0$), then it converges to the graded module $S_0(H^*(P))$ associated with the filtration (30) of $H^*(P)$.

In view of lemma 2.6, the following infinite chain of module inclusions

$$\Omega^* = \mathcal{C}^0\Omega^* \supset \mathcal{C}^1\Omega^* \supset \mathcal{C}^2\Omega^* \supset \cdots \supset \mathcal{C}^p\Omega^* \supset \cdots \tag{31}$$

is a filtered complex.

Definition 4.1. The above filtered complex (31) is said to be the $\mathcal{C}$*-filtration*. The induced spectral sequence is said to be the $\mathcal{C}$*-spectral sequence*.

We recall that, from the definition of spectral sequence associated with a filtered complex, the first term $(S_0^{p,q}, s_0)$ of the $\mathcal{C}$-spectral sequence is just the graded module associated with the $\mathcal{C}$-filtration, *i.e.*,

$$S_0^{p,q} = \mathcal{C}^p\Omega^{p+q} / \mathcal{C}^{p+1}\Omega^{p+q},$$

with differential $s_0 \colon S_0^{p,q} \to S_0^{p,q+1}$, $s_0([\alpha]) = [d\alpha]$. The $\mathcal{C}$-spectral sequence is a first quadrant spectral sequence, hence it converges to the graded group associated with the de Rham cohomology $H^*(J^\infty\pi)$ of the initial complex Ω^* of the $\mathcal{C}$-filtration. We stress that $H^*(J^\infty\pi)$ is filtered according with (30).

The proof of the following proposition is quite simple and derives from proposition 2.11 [18, 115, 116, 118, 121].

Proposition 4.2. *The horizontalization* $h^{p,q}$ *yields an isomorphism, denoted by the same symbol,*

$$h^{p,q} \colon S_0^{p,q} \to E_0^{p,q}, \quad [\alpha] \mapsto h^{p,q}(\alpha).$$

Moreover, the above isomorphism yields $s_0 = d_H$. It turns out that the first term of the $\mathcal{C}$-spectral sequence is just the family of complexes $(E_0^{p,*}, d_H)_{0 \le p < +\infty}$, or, equivalently, the family of columns of the diagram (20).

Remark 4.3. The reader may wonder about how to recover rows of the variational bicomplex within the $\mathcal{C}$-spectral sequence approach. There is another natural filtration of Ω^*: it is provided by horizontal forms. Namely, one could consider the ideal of forms generated by the codistribution $T^*\pi\colon T^*M \to T^*J^\infty\pi$ and its powers. This filtration is preserved by d and yields another spectral sequence, whose 0-term consists of the rows of the diagram (20).

The computations of the remaining terms of the $\mathcal{C}$-spectral sequence will be done in subsection 4.3.

4.2 Forms and differential operators

The computation of the $\mathcal{C}$-spectral sequence has been performed in [115, 116, 118] in the language of differential operators. More precisely, there is an isomorphism between the spaces $E_0^{p,q}$ and suitable spaces of differential operators. As a by-product, we will obtain a description of the spaces $E_1^{p,q}$. The purpose of this subsection is to recall the basic facts about differential operators and to state the above mentioned isomorphism.

We now recall the basic algebraic and geometric setting for differential operators. The interested reader could consult [18, 66, 121] for more details.

Let P, Q be modules over an algebra A over $\mathbb{R}$. We recall ([1]) that a *linear differential operator* of order k is defined to be an $\mathbb{R}$-linear map $\Delta : P \to Q$ such that

$$[\delta_{a_0}, [\dots, [\delta_{a_k}, \Delta] \dots]] = 0 \tag{32}$$

for all $a_0, \dots, a_k \in A$. Here, square brackets stand for commutators and δ_{a_i} is the multiplication morphism. Of course, linear differential operators of order zero are morphisms of modules. The A-module of differential operators of order k from P to Q is denoted by $\mathrm{Diff}_k(P, Q)$. The A-module of differential operators of any order from P to Q is denoted by $\mathrm{Diff}(P, Q)$. This definition can be generalized to maps between the product of the A-modules $P_1, \dots P_l$ and Q which are differential operators of order k in each argument, *i.e.*, *multidifferential operators*. The corresponding space is denoted by $\mathrm{Diff}_k(P_1, \dots, P_l; Q)$, or, if $P_1 = \dots = P_l = P$, by $\mathrm{Diff}_{(l)\,k}(P, Q)$. Accordingly, we define $\mathrm{Diff}_{(l)}(P, Q)$.

Let P, Q be modules of sections of a vector bundle over the same basis M, and suppose that $(e_i)_{0\leq i\leq p}$, $(f_j)_{1\leq j\leq q}$ are local bases for their respective sections. Then it can be proved that a differential operator $\Delta \in \mathrm{Diff}_k(P, Q)$ acts in coordinates as expected:

$$\Delta(s) = a^{j\boldsymbol{\sigma}}_i \frac{\partial^{|\boldsymbol{\sigma}|} s^i}{\partial x^{\sigma_1} \cdots \partial x^{\sigma_n}} f_j, \qquad 0 \leq |\boldsymbol{\sigma}| \leq k, \quad \text{for all } s \in P, \tag{33}$$

where we used the coordinate expression $s = s^i e_i$. The proof makes use of Taylor expansions of the coefficients s^i and of the property (32).

Consider the chain of algebrae $\cdots \subset \mathcal{F}_k \subset \mathcal{F}_{k+1} \subset \cdots$, and two chains of modules of sections of vector bundles $\cdots \subset P_k \subset P_{k+1} \subset \cdots$ and $\cdots \subset Q_k \subset Q_{k+1} \subset \cdots$ over the previous algebrae, with direct limits P and Q. Then a differential operator $\Delta\colon P \to Q$ is an $\mathbb{R}$-linear map such that for all k the restriction $\Delta|_{P_k}$ is a differential operator $\Delta|_{P_k}\colon P_k \to Q_{k+l}$, where l can depend on k.

We will mainly use differential operators whose expressions contain total derivatives instead of standard ones. To do that, we say a $\mathcal{F}$-module P to be *horizontal* if it is the module of sections of $\pi_\infty^* V \to J^\infty\pi$, where $V \to M$ is a vector bundle. Of course, P can be seen as the direct limit of the chain of modules of sections of $\pi_r^* V \to J^r\pi$. Then, we say a differential operator $\Delta\colon P \to Q$ (of order k) between two horizontal modules P and Q to be $\mathcal{C}$*-differential* if it can be restricted to the manifolds of the form $j^\infty s(M)$, where s is a section of π. In other words, Δ is a $\mathcal{C}$-differential operator if the equality $j^\infty s(M)^*(\varphi) = 0$, $\varphi \in P$, implies $j^\infty s(M)^*(\Delta(\varphi)) = 0$ for any section $s\colon M \to E$. In local coordinates, we have $\Delta = a^{j\boldsymbol{\sigma}}_i D_{\boldsymbol{\sigma}}$, where $a^{j\boldsymbol{\sigma}}_i \in \mathcal{F}$.

We denote by $\mathcal{C}\mathrm{Diff}_k(P,Q)$ the $\mathcal{F}$-module of $\mathcal{C}$-differential operators of order k from P to Q. We also introduce the $\mathcal{F}$-module $\mathcal{C}\mathrm{Diff}(P,Q)$ of differential operators from P to Q of any order. We can generalize the definition to multi-$\mathcal{C}$-differential operators. In particular, we will be interested to spaces of antisymmetric multi-$\mathcal{C}$-differential operators, which we denote by $\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(l)\,k}(P,Q)$. Analogously, we introduce $\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(l)}(P,Q)$.

Now, we consider the two horizontal modules $\varkappa$ (16) and $E_0^{0,q} = \overline{\Omega}^{0,q}$. For a proof of the following proposition, see [18].

Proposition 4.4. *We have the natural isomorphism*

$$E_0^{p,q} \to \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa, E_0^{0,q}), \qquad \alpha \mapsto \nabla_\alpha \tag{34}$$

where $\nabla_\alpha(\varphi_1,\dots,\varphi_p) = E_{\varphi_p} \lrcorner (\dots \lrcorner (E_{\varphi_1} \lrcorner \alpha) \dots)$.

Note that the isomorphism holds because for any vertical tangent vector to $J^r\pi$ there exists an evolutionary field passing through it. In coordinates, if α

$$\alpha = \alpha^{\boldsymbol{\sigma}_1\cdots\boldsymbol{\sigma}_p}_{i_1\ \cdots i_p\ \lambda_1\cdots\lambda_q}\, \omega^{i_1}_{\boldsymbol{\sigma}_1} \wedge \cdots \wedge \omega^{i_p}_{\boldsymbol{\sigma}_p} \wedge dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_q}$$

then

$$\nabla_\alpha(\varphi_1,\dots,\varphi_p) = p!\alpha^{\boldsymbol{\sigma}_1\cdots\boldsymbol{\sigma}_p}_{i_1\ \cdots i_p\ \lambda_1\cdots\lambda_q}\, D_{\boldsymbol{\sigma}_1}\varphi^{i_1} \cdots D_{\boldsymbol{\sigma}_p}\varphi^{i_p}\, dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_q}$$

4.3 The $\mathcal{C}$-spectral sequence and its 1-st and 2-nd terms

The term $S_1^{p,q}$ of the $\mathcal{C}$-spectral sequence is computed in two steps. The following lemma yields the cohomology of the terms with $0 \leq q \leq n-1$. Let P be a horizontal module. Recall that $E_0^{0,q} = \overline{\Omega}^{0,q}$, the space of horizontal forms, and $E_0^{0,0} = \mathcal{F}$. We introduce the *adjoint module* $P^* \stackrel{\text{def}}{=} \mathrm{Hom}(P, E_0^{0,n})$. Consider the complex

$$0 \longrightarrow \mathcal{C}\mathrm{Diff}_{(p)}(P, E_0^{0,0}) \xrightarrow{w} \mathcal{C}\mathrm{Diff}_{(p)}(P, E_0^{0,1}) \cdots\cdots \mathcal{C}\mathrm{Diff}_{(p)}(P, E_0^{0,n}) \longrightarrow 0 \tag{35}$$

where the maps w are defined by $w(\nabla) \stackrel{\text{def}}{=} d_H \circ \nabla$.

Theorem 4.5. *The cohomology of the complex* (35) *is zero at* $\mathcal{C}\mathrm{Diff}_{(p)}(P, E_0^{0,q})$ *for* $0 \leq q \leq n-1$ *and is* $\mathcal{C}\mathrm{Diff}_{(p-1)}(P, P^*)$ *for* $q = n$.

The first version of the above theorem appeared in [116] (corollary 2; see also the longer paper [117]. The proof was published later [118] and used Spencer cohomology. Another proof based on the Koszul complex appeared in [106], but the formulation involved only differential forms in $E_0^{p,q}$. In the language of differential operators (see proposition 4.4), this amounts at considering the subspace $\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(P, E_0^{0,q}) \subset \mathcal{C}\mathrm{Diff}_{(p)}(P, E_0^{0,q})$, with $P = \varkappa$. The statement of the above theorem 4.5 is taken from [18, p. 190], but the proof is essentially the same as in [106]. Note that there is an obvious inclusion

$$\mathcal{C}\mathrm{Diff}_{(p-1)}(P, P^*) \subset \mathcal{C}\mathrm{Diff}_{(p)}(P, E_0^{0,n}).$$

There is an action of the permutation group S_p of p elements on $\mathcal{C}\mathrm{Diff}_{(p)}(P, E_0^{0,q})$. Namely, if $\tau \in S_p$ and $\nabla \in \mathcal{C}\mathrm{Diff}_{(p)}(P, E_0^{0,q})$ then for all $s_1, \ldots, s_p \in P$ we set $\tau(\nabla)(s_1, \ldots, s_p) \stackrel{\text{def}}{=} \nabla(s_{\tau(1)}, \ldots, s_{\tau(p)})$. This action commutes with w, so that we have the following corollary.

Corollary 4.6. *The skew-symmetric part of the complex* (35) *has zero cohomology at* $\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(P, E_0^{0,q})$ *for* $0 \leq q \leq n-1$.

It is easy to realize through the isomorphism of proposition 4.4 that, if $P = \varkappa$, then $w = d_H$ up to the isomorphism 34. Another set of terms of the $\mathcal{C}$-spectral sequence follows.

Corollary 4.7. *We have:*

- $S_1^{p,q} = 0$ *for* $p > 0$ *and* $0 \leq q \leq n-1$;
- $S_1^{0,q} = H^q(E)$ *for* $0 \leq q \leq n-1$.

The content of the above theorem is the same of theorem 3.14 and corollary 3.15. The proof of the second statement of above corollary follows from the convergence of the $\mathcal{C}$-spectral sequence to the de Rham cohomology. Indeed it can be quickly realized that the differential s_1 is the zero map on $S_1^{0,q}$ for $0 \le q \le n-1$ and that s_1 is never $S_1^{0,q}$-valued for $0 \le q \le n-1$. It follows that $S_2^{0,q} = S_\infty^{0,q}$ for $0 \le q \le n-1$. Moreover, we note that there is at most one nonzero term among $S_\infty^{p,q}$ with $p+q=k$. Then there exists a $\tilde{p}$ such that $F^{\tilde{p}}H^{\tilde{p}+q}(\Omega_r^*) \neq F^{\tilde{p}+1}H^{\tilde{p}+q}(\Omega_r^*)$. This implies that the filtration of the de Rham cohomology of the initial complex (Ω^*, d) is trivial:

$$H^{\tilde{p}+q}(\Omega_r^*) = F^0 H^{\tilde{p}+q}(\Omega_r^*) = \cdots = F^{\tilde{p}} H^{\tilde{p}+q}(\Omega_r^*) \supset 0 \cdots \supset 0, \tag{36}$$

whence $S_\infty^{\tilde{p},q} = H^{\tilde{p}+q}(\Omega_r^*)$.

We now calculate the last set of terms of E_1. The following elementary lemma comes directly from the definition of spectral sequence and the isomorphism (34).

Lemma 4.8. *We have*

$$S_1^{p,n} = E_0^{p,n}/d_H(E_0^{p,n-1}) = \mathcal{C}^p\Omega^p \wedge \overline{\Omega}^{0,n}/d_H(\mathcal{C}^p\Omega^p \wedge \overline{\Omega}^{0,n-1}) = E_1^{p,n}.$$

It turns out that

$$E_1^{p,n} \simeq \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa, E_0^{0,n})/d_H(\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa, E_0^{0,n-1})).$$

In view of the discussion preceding corollary 4.6, the term $E_1^{p,n}$ is isomorphic to the subspace $K_p(\varkappa) \subset \mathcal{C}\mathrm{Diff}_{(p-1)}(\varkappa, \varkappa^*)$ of elements which are invariant under the action of the permutation group S_p. To do that we need the notion of adjoint operator. Let P, Q be horizontal modules and $\Delta\colon P \to Q$ a $\mathcal{C}$-differential operator. Then Δ induces a map

$$\Delta'\colon \mathcal{C}\mathrm{Diff}(Q, E_0^{0,q}) \to \mathcal{C}\mathrm{Diff}(P, E_0^{0,q}), \quad \Delta'(\nabla) = \nabla \circ \Delta.$$

The map Δ' is a cochain map for the complex (35) (with $p=1$), in the sense that $\Delta' \circ w = w \circ \Delta'$. Hence Δ' yields a cohomology map which, according to theorem 4.5, is trivial if $0 \le q \le n-1$ and is denoted by $\Delta^*\colon Q^* \to P^*$ if $q = n$.

Definition 4.9. The operator Δ^* is said to be the *adjoint operator* to Δ.

In coordinates, following the same notation of (33), we have $\Delta = a_i^{j\boldsymbol{\sigma}} D_{\boldsymbol{\sigma}}$. If $\{e^i \otimes \nu\}$, $\{f^j \otimes \nu\}$ are two local bases respectively of P^* and Q^*, and $s^* \in P^*$, $t^* \in Q^*$, we have $s^* = s_i e^i \otimes \nu$, $t^* = t_j f^j \otimes \nu$, and

$$\Delta^*(t^*) = (-1)^{|\boldsymbol{\sigma}|} D_{\boldsymbol{\sigma}}(a_i^{j\boldsymbol{\sigma}} t_j) e^i \otimes \nu. \tag{37}$$

In fact, it can be easily proved that the composition $\nabla \circ \Delta$ is locally equal to the above expression up to an operator in $\operatorname{im} w$. Of course, locally this is just integration by parts. The global meaning of the expression (37) appears in the two following statements (for a proof, see [18, 66]). If $\Delta \in \mathcal{C}\mathrm{Diff}_{(p)}(P, Q)$, then for any $p_1, \ldots, p_{p-1}$ we define $\Delta(p_1, \ldots, p_{p-1}) \in \mathcal{C}\mathrm{Diff}(P, Q)$ in the following obvious way:

$$\Delta(p_1, \ldots, p_{p-1})(p_p) = \Delta(p_1, \ldots, p_{p-1}, p_p). \tag{38}$$

Next lemma shows how to determine the representative of each n-th cohomology class of the complex (35).

Lemma 4.10. *Let P be a horizontal module and $\Delta \in \mathcal{C}\mathrm{Diff}_{(p)}(P, E_0^{0,n})$. Then*

$$\Delta(p_1, \ldots, p_{p-1}) = \Delta(p_1, \ldots, p_{p-1})^*(1) + w(\nabla(p_1, \ldots, p_{p-1})), \tag{39}$$

where $\nabla(p_1, \ldots, p_{p-1}) \in \mathcal{C}\mathrm{Diff}(P, E_0^{0,n-1})$. It turns out that $w(\nabla(p_1, \ldots, p_{p-1})) = w(\tilde{\nabla})$, with $\tilde{\nabla} \in \mathcal{C}\mathrm{Diff}_{(p)}(P, E_0^{0,n})$.

The proof is achieved first locally, with a relatively easy computation, then globally by observing that $\Delta(p_1, \ldots, p_{p-1})^*(1)$ is a natural operator and the representative of a cohomology class, hence the difference $\Delta(p_1, \ldots, p_{p-1}) - \Delta(p_1, \ldots, p_{p-1})^*(1)$ must lie in the image of w. See also [18, 66].

The above operator $w(\tilde{\nabla})$ is uniquely determined, but $\tilde{\nabla}$ is not. The problem of determining under which additional requirements $\tilde{\nabla}$ is uniquely determined has been thoroughly analysed in [2, 4, 61] (see remark 3.6). Eq. (39) is a consequence of the fact that every object in the n-th cohomology class of the complex (35) is globally represented by a single homomorphism in P^*.

Proposition 4.11 (Green's formula)**.** *Let P, Q be horizontal modules and $\Delta \colon P \to Q$ a $\mathcal{C}$-differential operator. Then*

$$q^*(\Delta(p)) - (\Delta^*(q^*))(p) = d_H(\omega_{p,q^*}(\Delta)) \tag{40}$$

for all $q^* \in Q^*$, $p \in P$, where $\omega_{p,q^*}(\Delta) \in E_0^{0,n-1}$ and $\omega_{p,q^*}(\Delta)$ is a $\mathcal{C}$-differential operator with respect to p and q^*.

The above formula has been introduced in [115] (but see also [18, 66, 118]); its proof is a simple consequence of lemma 4.10.

Now, it is easy to see that the action of a permutation of the first $p-1$ arguments of $\square \in \mathcal{C}\mathrm{Diff}_{(p-1)}(\varkappa, \varkappa^*)$ commutes with the splitting of lemma 4.10, hence $K_p(\varkappa) \subset \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p-1)}(\varkappa, \varkappa^*)$. Then, for $\Delta \in \mathcal{C}\mathrm{Diff}_{(p-1)}(\varkappa, \varkappa^*)$ and for any $p_1, \ldots, p_p$ we define $\Delta_j(p_1, \ldots, \widehat{p_j}, \ldots, p_{p-1}) \in \mathcal{C}\mathrm{Diff}(\varkappa, \varkappa^*)$ in the following obvious way:

$$\Delta_j(p_1, \ldots, \widehat{p_j}, \ldots, p_{p-1})(p_j)(p_p) = \Delta(p_1, \ldots, p_{p-1})(p_p). \tag{41}$$

Due to Green's formula we have

$$\Delta_j(p_j)(p_p) = \Delta_j^*(p_p)(p_j) + d_H(\omega_{p,q^*}(\Delta)).$$

This implies that $K_p(\varkappa) \subset \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p-1)}(\varkappa, \varkappa^*)$ is the subset of skew-adjoint operators with respect to the exchange of one of the first $p-1$ arguments with the last one. Hence, we proved the following theorem.

Theorem 4.12. *There is an isomorphism*

$$I\colon E_1^{p,n} \to K_p(\varkappa), \quad [\Delta] \mapsto \Delta^*(1), \tag{42}$$

where the adjoint is taken with respect to one of the arguments of Δ.

Let us see the coordinate expression of I in the most meaningful cases. We will represent elements of $E_0^{p,n}$ through the isomorphism 34. We set $\nu \stackrel{\mathrm{def}}{=} dx^1 \wedge \cdots \wedge dx^n$.

Case $p = 1$**:** let $[\alpha] \in E_1^{1,n}$. Then $\nabla_\alpha(\varphi) = \alpha_i^{\boldsymbol{\sigma}}\, D_{\boldsymbol{\sigma}} \varphi^i\, \nu$ and

$$I([\alpha])(\varphi) = (-1)^{|\boldsymbol{\sigma}|} D_{\boldsymbol{\sigma}} \alpha_i^{\boldsymbol{\sigma}}\, \varphi^i\, \nu.$$

Considerations similar to what exposed in section 3.1 apply also here.

Case $p = 2$**:** let $[\alpha] \in E_1^{1,n}$. Then $\nabla_\alpha(\varphi_1, \varphi_2) = 2\alpha_{i\ j}^{\boldsymbol{\sigma}\boldsymbol{\tau}} D_{\boldsymbol{\sigma}} \varphi_1^i D_{\boldsymbol{\tau}} \varphi_2^j\, \nu$ (with $\alpha_{i\ j}^{\boldsymbol{\sigma}\boldsymbol{\tau}} = -\alpha_{j\ i}^{\boldsymbol{\tau}\boldsymbol{\sigma}}$) and, if α is a form on the r-th order jet, then

$$\begin{aligned} I([\alpha])(\varphi_1)(\varphi_2) &= (-1)^{|\boldsymbol{\tau}|} D_{\boldsymbol{\tau}} (2\alpha_{i\ j}^{\boldsymbol{\sigma}\boldsymbol{\tau}} D_{\boldsymbol{\sigma}} \varphi_1^i) \varphi_2^j\, \nu \\ &= \sum_{\substack{\boldsymbol{\mu}+\boldsymbol{\sigma}=\boldsymbol{\rho} \\ 0 \le |\boldsymbol{\rho}| \le 2r}} (-1)^{|\boldsymbol{\xi}+\boldsymbol{\mu}|} \frac{(\boldsymbol{\xi}+\boldsymbol{\mu})!}{\boldsymbol{\xi}!\boldsymbol{\mu}!} 2 D_{\boldsymbol{\xi}} \alpha_{ij}^{\boldsymbol{\sigma}\boldsymbol{\xi}+\boldsymbol{\mu}} D_{\boldsymbol{\rho}} \varphi_1^i \varphi_2^j\, \nu. \end{aligned} \tag{43}$$

Note that the above expressions coincide with the expressions of subsection 3.1 up to the isomorphism (34) and a constant factor (which depends on different conventions about numerical factors and the ordering in contractions and wedge products).

Remark 4.13. We observe that in [116] an intrinsic expression of e_1 which makes use of the above isomorphism was provided (see also [18, p. 195–197]).

A last step is needed in order to complete the computation of the $\mathcal{C}$-spectral sequence.

Theorem 4.14. *We have:*

- $S_2^{0,q} = H^q(E)$ *for* $0 \leq q \leq n-1$*;*
- $S_2^{p,q} = 0$ *for* $p > 0$ *and* $0 \leq q \leq n-1$*;*
- $S_2^{p,n} = H^{p+n}(E)$ *for* $0 \leq p$*.*

It turns out that $S_2 = S_\infty$*.*

The only non-trivial statement of the above theorem is the last one. This follows from the convergence of the $\mathcal{C}$-spectral sequence to the de Rham cohomology and the fact that the differential e_2 always point either from 0 to $S_2^{p,n}$ or from $S_2^{p,n}$, so that it is the trivial map in both cases. For more details, see [18, 66, 118]. We just recall that the computation of the $\mathcal{C}$-spectral sequence for $J^\infty\pi$ is called *one-line theorem* [118, 121].

Corollary 4.15. *The variational sequence is obtained from the* $\mathcal{C}$*-spectral sequence by joining the two complexes* $(E_0^{0,q}, d_H)$ *and* $(E_1^{p,n}, e_1)$*.*

The above corollary is proved after proving that $s_1 = e_1$. This is quite easy, see [124].

We stress once again that the above construction yields the same results as in section 3 with the only exception of the cohomology of the rows. For this another spectral sequence would produce the results, namely the one arising from a filtration through horizontal forms.

5 Finite order variational sequence

The variational bicomplex and its derivation through the spectral sequence have been derived so far on infinite order jets. The reasons for doing that have been explained in section 2. But both the variational bicomplex and its derivation through the spectral sequence admit a finite-order counterpart, which has been studied in recent years.

The first statement of a partial version of finite order variational sequence was in [5]. This finite order variational sequence stopped with a trivial projection to 0 just after the space of finite order source forms (see section 3.1). The local exactness of this sequence was proved, together with an original solution of the global inverse problem (despite the fact that in order to do that the authors used infinite order jets). For more detailed comments about that variational sequence see remark 5.7.

The first formulation of a (long) variational sequence on finite order jet spaces is in [70] (see [72] for the case $n = 1$). Below we will describe the main points of the approach of [70], and compare it with other approaches. We also observe that more details can be found in [74]. The $\mathcal{C}$-spectral sequence on finite order jets of fibrings has been recently computed; the interested reader can find it in [125].

For the sake of completeness we also mention the paper [46]. In that paper the exactness of the horizontal de Rham sequence on finite order jets of submanifolds is proved. Nonetheless, we stress that this result could also be easily derived from the exactness results in [5, 70]. Another contribution has been given in [95], where the author stresses the relationship between a part of the finite order variational sequence and the Spencer sequence. This relationship was already explored in [115, 116] in the case of infinite order jet spaces.

The scheme of the finite order approach of [70] is the following. First of all we stress that the approach is developed in the language of sheaves. In [70] a natural exact subsequence of the de Rham sequence on $J^r\pi$ is defined. This subsequence is made by contact forms and their differentials. Then we define the r–th order variational sequence to be the quotient of the de Rham sequence on $J^r\pi$ by means of the above exact subsequence. Local and global results about the variational sequence are proved using the fact that the above subsequence is globally exact and using the abstract de Rham theorem.

Let us consider the sheaf of 1-contact forms $\mathcal{C}^1\Omega^*$, and denote by $(d\mathcal{C}^1\Omega^k)\breve{}$ the sheaf generated by the presheaf $d\mathcal{C}^1\Omega^k$. We set

$$\begin{aligned} &\Theta^q_r \overset{\text{def}}{=} \mathcal{C}^1\Omega^q_r + (d\mathcal{C}^1\Omega^{q-1}_r)\breve{} \quad 0 \leq q \leq n, \\ &\Theta^{p+n}_r \overset{\text{def}}{=} \mathcal{C}^p\Omega^{p+n}_r + (d\mathcal{C}^p\Omega^{p+n-1}_r)\breve{} \quad 1 \leq p \leq \dim J^r\pi. \end{aligned} \tag{44}$$

We observe that $d\mathcal{C}^1\Omega^{q-1}_r \subset \mathcal{C}^1\Omega^q_r$, so that the second summand of the above first equation yields no contribution to $\mathcal{C}^1\Omega^q_r$. Moreover, we observe that $\Theta^{p+n}_r = 0$ if $p + n > P$, where the value of P (which is strictly less than $\dim J^r\pi$) is explicitly computed in [70], and depends on the results of Theorem 2.4. We also have the following property (proved in [70]).

Lemma 5.1. *Let $0 \leq k \leq \dim J^r\pi$. Then the sheaves Θ^k_r are soft sheaves.*

We have the following natural soft subsequence of the de Rham sequence on $J^r\pi$

$$0 \longrightarrow \Theta^1_r \xrightarrow{d} \Theta^2_r \xrightarrow{d} \cdots \xrightarrow{d} \Theta^P_r \xrightarrow{d} 0 \tag{45}$$

Definition 5.2. The sheaf sequence (45) is said to be the *contact sequence.*

Theorem 5.3. *The contact sequence is an exact soft resolution of $\mathcal{C}^1\Omega^1_r$, hence the cohomology of the associated cochain complex of sections on any open subset of $J^r\pi$ vanishes.*

The above theorem is proved in [70] by first proving the local exactness of the contact sequence and then using standard results from sheaf theory (for which an adequate source is [126]).

Standard arguments of homological algebra prove that the following diagram is commutative, and its rows and columns are exact.

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & & \\
 & & 0 \arrow[r] & \Theta^1_r \arrow[r, "d"] \arrow[d] & \Theta^2_r \arrow[r, "d"] \arrow[d] & \cdots \arrow[r, "d"] & \Theta^P_r \arrow[r, "d"] \arrow[d] & 0 & \\
0 \arrow[r] & \mathbb{R} \arrow[r] & \Omega^0_r \arrow[r, "d"] \arrow[dr] & \Omega^1_r \arrow[r, "d"] \arrow[d] & \Omega^2_r \arrow[r, "d"] \arrow[d] & \cdots \arrow[r, "d"] & \Omega^P_r \arrow[r, "d"] \arrow[d] & \Omega^{P+1}_r \arrow[r, "d"] & \cdots 0 \\
 & & & \Omega^1_r/\Theta^1_r \arrow[r, "\mathcal{E}_1"] \arrow[d] & \Omega^2_r/\Theta^2_r \arrow[r, "\mathcal{E}_2"] \arrow[d] & \cdots \arrow[r, "\mathcal{E}_{P-1}"] & \Omega^P_r/\Theta^P_r \arrow[ur, "\mathcal{E}_P"] \arrow[d] & & \\
 & & & 0 & 0 & & 0 & &
\end{tikzcd}
$$

Definition 5.4. The above diagram is said to be the *r-th order variational bicomplex* associated with the fibred manifold $\pi\colon E \to M$. We say the bottom row of the above diagram to be the *r-th order variational sequence* associated with the fibred manifold $\pi\colon E \to M$.

Due to theorem 5.3 the finite order variational sequence is an exact sheaf sequence (this means that the sequence is locally exact, [126]). Hence both the de Rham sequence and the variational sequence are acyclic resolutions of the constant sheaf $\mathbb{R}$ ('acyclic' means that the sequences are locally exact with the exception of the first sheaf $\mathbb{R}$). Next corollary follows at once.

Corollary 5.5. *The cohomology of the variational sequence is naturally isomorphic to the de Rham cohomology of $J^r\pi$.*

Having already dealt with local and global properties of the r-th order variational sequence, we are left with the problem of representing the quotient sheaves. Now it is obvious that, for $0 \leq q \leq n$, horizontalization provides such a representation (see [70, 122]).

Proposition 5.6. *Let $0 \leq q \leq n$. Then we have the isomorphism*

$$J_q \colon \Omega^q_r/\Theta^q_r \to \overline{\Omega}^{0,q}_r, \quad [\alpha] \mapsto h^{0,q}(\alpha).$$

The quotient differential $\mathcal{E}_q$ reads through the above isomorphism as

$$J_{q+1}(\mathcal{E}_q([\alpha])) = J_{q+1}([d\alpha]) = h^{0,q+1}(d\alpha) = d_H h^{0,q+1}(\alpha).$$

The last equality of the above equation is the least obvious, and was first proved in [5]. The proof depends on the fact that $D_\lambda u^i_{\boldsymbol{\sigma}\mu} = u^i_{\boldsymbol{\sigma}\mu\lambda}$, and that the indexes λ, μ are skew-symmetrized in the coefficients of α (see the coordinate expression of $h^{0,q}$).

Remark 5.7. In [5] the finite order variational sequence is developed starting from the idea of finding a subsequence of forms whose order do not change under d_H. The authors prove that the above property characterizes the forms which are the image of $h^{0,q}$ (see also [4]). Conversely, in [70] the idea is to start with forms on finite order jets, but the result is the same up to the degree $q = n$.

When the degree of forms is greater than n we are able to provide isomorphisms of the quotient sheaves with other quotient sheaves made with proper subsheaves. This helps both to the purpose of representing quotient sheaves and to the purpose of comparing the current approach with others, as we will see.

Proposition 5.8. *The horizontalization $h^{p,n}$ induces the natural sheaf isomorphism*

$$J_{p+n} \colon \Omega^{p+n}_r/\Theta^{p+n}_r \to \overline{\Omega}^{p,n}_r/h^{p,n}((d\mathcal{C}^p\Omega^{p+n-1}_r)\check{}), \quad [\alpha] \mapsto [h^{p,n}(\alpha)].$$

The quotient differential $\mathcal{E}_{p+n}$ reads through the above isomorphism as

$$J_{p+1+n}(\mathcal{E}_{p+n}([\alpha])) = J_{p+1+n}([d\alpha]) = [h^{p+1,n}(d\alpha)] = [d_V h^{p,n}(\alpha)].$$

For a proof, see [124]. A similar approach is in [67, 68].

Now, it is clear from proposition 5.8 that we are able to represent the quotient sheaves $\overline{\Omega}^{p,n}_r/h^{p,n}((d\mathcal{C}^p\Omega^{p+n-1}_r)\check{})$ using the interior Euler operator restricted to $\overline{\Omega}^{p,n}_r$; this is precisely the approach of [67, 68]. See [47] for a different approach to this problem. A further approach to the problem of representation appeared in [76]. Here the concept of *Lepagean equivalent* is introduced in full generality (older version of this concept can be found *e.g.*, in [69], with references to older foundational works). Namely, let $\alpha \in \Omega^{p+n}_r$.

Then a Lepage equivalent of $[\alpha] \in E_1^{p,n}$ is a differential form $\beta \in \Omega_r^{p+n}$ such that

$$h^{p,n}(\beta) = h^{p,n}(\alpha), \qquad h^{p+1,n}(d\beta) = I(h^{p+1,n}(d\alpha)) = e_1([\alpha]).$$

The most important example of a Lepagean equivalent is the Poincaré–Cartan form of a Lagrangian (see, *e.g.*, [74]). A representation of forms in the variational sequence through Lepagian equivalents is currently being studied also in exterior differential systems theories [23].

Remark 5.9. It is interesting to observe that, either in view of theorem 4.5 or in view of the results by several authors referred to in remark 3.6, every form $\alpha \in \overline{\Omega}_r^{p,n}$ can be written as the sum $\alpha = \sigma + d_H\gamma$, where σ can be seen either as a skew-adjoint differential operator (from the isomorphism of proposition 4.4 and theorem 4.12) or as a form in the image of the interior Euler operator (which admits an equivalent characterization as skew-adjoint form, see [4]).

This means that, despite the fact that the denominator in proposition 5.8 is made by forms which are *locally* total divergences, only global divergences really matter.

The finite order formulation of [70] yields a variational sequence which can be proved to be equal to the finite order variational sequence obtained from a finite order analogue of the $\mathcal{C}$-spectral sequence [125]. Moreover, as one could expect, for $0 \leq s < r$ pull-back via $\pi_{r,s}$ yields a natural inclusion of the s-th order variational bicomplex into the s-th order variational bicomplex. More precisely, we have the following lemma (see [70]).

Lemma 5.10. *Let $0 \leq s < r$. Then we have the injective sheaf morphism*

$$\chi_s^r : \left(\Omega_s^k/\Theta_s^k\right) \to \left(\Omega_r^k/\Theta_r^k\right), \quad [\alpha] \mapsto [\pi_{r,s}^*\alpha].$$

Hence, there is an inclusion of the s–th order variational bicomplex into the r–th order variational bicomplex.

It can be proved that there exists an infinite order analogue of the above r-th order variational bicomplex [123]. This is defined in view of the above lemma via a direct limit of the injective family of r-th order variational bicomplexes. Nonetheless the direct limit infinite order bicomplex will be a bicomplex of presheaves, because gluing forms defined on jets of increasing order provides ‘forms’ which are only locally of finite order (see [43, 123] and the comments after theorem 3.14).

Remark 5.11. The main motivation for the finite order variational sequence has been a refinement in inverse problems of the calculus of variations. For example, a source form $[\alpha] \in \Omega^{n+1}_r/\Theta^{n+1}_r$ which is locally variational, *i.e.* $e_1([\alpha]) = 0$, admits a (local) Lagrangian $[\beta] \in \Omega^n_r/\Theta^n_r$. A representative of $[\beta]$ is $h^{0,n}(\beta)$, which is defined on the $r+1$-st order jet and depends on highest order derivatives through hyperjacobians (proposition 5.6). See [73, 123, 124, 125] for a comparison between the finite order and infinite order approaches.

6 Special topics

Due to space and time constraints it is not possible to go further in describing in details the current achievements in variational sequence theory. It is also impossible to reserve to applications and examples more than just a mention. The above tasks would require writing a whole book. But in this section at least the most important research directions of the last 15 years will be exposed, with reference to the literature for the readers who are interested in knowing more.

6.1 Inverse problem of the calculus of variations

The variational sequence is intimately related with the inverse problem of the calculus of variations (see the Introduction). This problem has a long history for which possible sources are the notes [91, p. 377] and references quoted therein, and [14, 88, 78] for the case of mechanics ($n = 1$). Here we briefly describe some inverse problems arising in the variational sequence, including the inverse problem of the calculus of variations. We just recall that the cohomology of the de Rham sequence on E is isomorphic to the cohomology of the variational sequence.

Variationally trivial Lagrangians. A variationally trivial Lagrangian is an element $[\alpha] \in E^{0,n}_1$ such that $e_1([\alpha]) = 0$. If $[\alpha]$ is a variationally trivial Lagrangian, then $[\alpha]$ is locally a total divergence, *i.e.*, $[\alpha] = d_H[\beta]$ with $[\beta] \in E^{0,n-1}_0$. A global horizontal $n-1$-form $[\beta] \in E^{0,n-1}_0$ such that $[\alpha] = d_H[\beta]$ exists if and only if $[[\alpha]] = 0 \in H^n(E)$. A refinement of this result is the following theorem.

Theorem 6.1. *Let $\lambda \colon J^r\pi \to \wedge^n T^*M$ induce a variationally trivial Lagrangian $[\lambda]$. Then, locally, $\lambda = d_H\mu$, where $\mu = h^{0,n-1}(\alpha)$ and $\alpha \in \Omega^{n-1}_{r-1}$.*

In other words, according to the above hypotheses, $\lambda = h^{0,n}(d\alpha)$, hence it depends on r-th order derivatives through hyperjacobians. This result has been proved in [5, 15, 75, 48][6] using various techniques. Note that the result is better with respect to the order of jets than what can be obtained by the local exactness of the finite order variational sequence. In fact, from the finite order variational sequence we would obtain $\alpha \in \Omega^{n-1}_r$. Of course, the result is sharp: the order cannot be further lowered.

Locally variational source forms. A locally variational source form is an element $[\alpha] \in E_1^{1,n}$ such that $e_1([\alpha]) = 0$. If $[\alpha]$ is a locally variational source form, then $[\alpha]$ is locally the Euler–Lagrange expression of a (local) Lagrangian, *i.e.*, $[\alpha] = \mathcal{E}[\beta]$ with $[\beta] \in E_1^{0,n}$. A global Lagrangian $[\beta] \in E_0^{0,n}$ such that $[\alpha] = \mathcal{E}[\beta]$ exists if and only if $[[\alpha]] = 0 \in H^{n+1}(E)$. A refinement of this result, like in the previous inverse problem, is much more difficult. We list the results which have been achieved till now.

Theorem 6.2. *Let $[\alpha] \in \Omega^{n+1}_r/\Theta^{n+1}_r$ be locally variational. Then there exists a (local) Lagrangian $[\beta] \in \Omega^n_r/\Theta^n_r$ such that $[\alpha] = \mathcal{E}[\beta]$.*

The above result is a direct consequence of the local exactness of the finite order variational sequence, and, as before, it is sharp with respect to the order [70, 122]. However, it can be very difficult to check that a source form is in the space $\Omega^{n+1}_r/\Theta^{n+1}_r$. A result proved in [4] is helpful in this sense. Let $u^{(r)}$ denote all derivative coordinates of order r on a jet space. Let $f \in C^\infty(J^{2r}\pi)$, and suppose that

$$f(x^\lambda, u^{(0)}, \dots, u^{(r)}, tu^{(r+1)}, t^2u^{(r+2)}, \dots, t^r u^{(2r)})$$

is a polynomial of degree less than or equal to r in $u^{(s)}$, with $r+1 \leq s \leq 2r$. Then f is said to be a weighted polynomial of degree r in the derivative coordinates of order $r+1 \leq s \leq 2r$.

Theorem 6.3. *Let $[\Delta]$ be a locally variational source form induced by $\Delta \colon J^{2r}\pi \to C_0^* \wedge \wedge^n T^*M$. Suppose that the coefficients of Δ are weighted polynomials of degree less than or equal to r. Then $\Delta = \mathcal{E}(\lambda)$, where $\lambda \colon J^r\pi \to \wedge^n T^*M$.*

Again, the result is sharp with respect to the order of the jet space where the Lagrangian is defined. The above theorem is complemented in [4] by a rather complex algorithm for building the lowest order Lagrangian. This algorithm is an improvement of the well-known Volterra Lagrangian

$$L = \int_0^1 u^i \Delta_i(x^\lambda, tu^j_{\boldsymbol{\sigma}})dt$$

[6]In [15] the proof is for the special case when the Lagrangian does not depend on (x^λ).

for a locally variational source form Δ. In fact, the above Lagrangian is defined on the same jet space as Δ. The finite order variational sequence yields another method for computing lower order Lagrangians, provided we know that $\Delta = [\alpha] \in \Omega^{n+1}_r/\Theta^{n+1}_r$. Namely, we apply the contact homotopy operator to the closed form $d\alpha \in \Theta^{n+2}_r$, finding $\beta \in \Theta^{n+1}_r$ such that $d\beta = d\alpha$. By using once again using the (standard) homotopy operator we find $\gamma \in \Omega^n_r$ such that $d\gamma = \beta - \alpha$, and $\lambda \overset{\text{def}}{=} h^{0,n}(\gamma)$ is the required Lagrangian. Of course, the most difficult point is to invert the representation of quotients in the variational sequence, *i.e.*, to find a least order α such that $\Delta = [\alpha]$.

The above theorem does not exhaust the finite order inverse problem. A locally variational source form on $J^{2r}\pi$ seems to have a definite form of the coefficients with respect to its derivatives of order s, with $r+1 \leq s \leq 2r$. A conjecture in this sense is formulated in [4] in an admittedly imprecise way. We *conjecture* that locally variational source forms defined on $J^{2r}\pi$ could be elements of $\Omega^{n+1}_r/\Theta^{n+1}_r$. Note that the representation through I of elements in $\Omega^{n+1}_r/\Theta^{n+1}_r$ yields source forms which are of order $2r+1$ and are obtained through the adjoint of the horizontalization of a form in Ω^{n+1}_r (which is a hyperjacobian polynomial of degree at most n in derivatives of order r); see [122] for more details about the structure of such forms.

Finally, we recall that recently some geometric results on variational first-order partial differential equations have been obtained in [54]. Such equations arise in multisymplectic field theories.

Symplectic structures. In [33] the symplectic structures for evolution equations are introduced. They are dual to the Hamiltonian structures mentioned in the introduction. A symplectic structure is an element $[\alpha] \in E^{2,n}_1$ such that $e_1(\alpha) = 0$ (see also[18]). It is clear that another inverse problem arises here. But there are no results as on the above section. It seems natural to formulate a conjecture on the structure of symplectic structures by analogy with the above conjecture.

Variational problems defined by local data. There are some examples of global source forms which do not admit a global Lagrangian. For instance, Galilean relativistic mechanics [97] and Chern–Simons field theories (where a global Lagrangian indeed exists but it is not gauge-invariant). Some authors proposed a general formalism for dealing with such situations. Namely, they introduce a sheaf of local n-forms all of which produce the same Euler–Lagrange source form under the action of $\mathcal{E}$. See [21, 19, 103, 104] for more details.

6.2 Variational sequence on jets of submanifolds

Let E be an $n+m$-dimensional manifold, and $x \in E$. We say that two n-dimensional submanifolds L_1, L_2 such that $x \in L_1 \cap L_2$ are *r-equivalent* if they have a contact of order r at x. It is possible to choose a chart of E at x of the form (x^λ, u^i), $1 \leq \lambda \leq n$, $1 \leq i \leq m$, where both L_1 and L_2 can be expressed as graphs $u^i = f_1^i(x^\lambda)$, $u^i = f_2^i(x^\lambda)$. Then the contact condition is the equality of the derivatives of the above functions at x up to the order r. This is an equivalence relation whose quotient set is $J^r(E,n)$, the *r-th order jet space of n-dimensional submanifolds of E*[7]. If E is endowed with a fibring π, then $J^r\pi$ is the open and dense subspace of $J^r(E,n)$ which is made by submanifolds which are transverse to the fibring at a point (which, of course, can be locally identified with the images of sections, hence with local sections themselves).

Of course, jets of submanifolds have a contact distribution, hence a $\mathcal{C}$-spectral sequence can be formulated [29, 115, 116]. As a by-product a variational sequence is obtained. Jets of submanifolds can also be seen as jets of parametrizations of submanifolds (*i.e.*, jets of local n-dimensional immersions) up to the action of the reparametrization group [63]. In this setting another approach to the variational sequence is [99]. In [84] the finite-order $\mathcal{C}$-spectral sequence on jets of submanifolds is computed. See also the more comprehensive treaties [1, 119, 121] on the geometry of jets of submanifolds, partial differential equations and the calculus of variations. Another approach to the calculus of variations on jets of submanifolds can be found in [49].

6.3 Variational sequence on differential equations

There are several books on the geometric theory of differential equations (see the Introduction). We invite the interested reader to consult them. Here we just recall the main result related to the variational sequence on differential equations.

A differential equation (ordinary or partial, scalar or system) is a submanifold $S \subset J^r(E,n)$. Such a submanifold inherits the contact distribution from $J^r(E,n)$, hence the $\mathcal{C}$-spectral sequence can be defined on it. Let us describe what are the main differences with the 'trivial equation case', *i.e.*, the case of $S = J^r(E,n)$ or $S = J^r\pi$.

First of all, we observe that the term $E_1^{0,n-1}$ of the $\mathcal{C}$-spectral sequence of an equation is made by equivalence classes of conservation laws of the given

[7]The synonyms 'manifold of contact elements' [28] and 'extended jet bundles' [91] are also used.

equation up to trivial conservation laws. To realize it, it is sufficient to recall that conservation laws take the form of a total divergence which vanishes on the given equation (like, *e.g.*, continuity equations).

If S is closed then it can be represented as $F = 0$, where F is a section of a vector bundle over $J^r(E, n)$. Any differential equation $S = S^{(0)} \subset J^r(E, n)$ can be prolonged to a differential equation $S^{(1)} \subset J^{r+1}(E, n)$ which is locally described as $D_\lambda F^i = 0$. By iterating this procedure we obtain a sequence $\{S^{(i)}\}_{0\leq i\leq +\infty}$. We require that the equation S be *formally integrable*: this amounts at requiring that for every $i \in \mathbb{N}$ the restriction of $\pi_{i+1,i}$ to $S^{(i+1)}$ be a bundle over $S^{(i)}$. Hence the inverse limit $S^{(\infty)}$ can be constructed. We also require that the equation be *regular*: this means that the ideal of functions on $S^{(\infty)}$ is functionally generated by the differential consequences $D_{\boldsymbol{\sigma}} F^i$ of F. Finally, we say that S is ℓ-normal if the linearization of F has maximal rank (see [18, p. 198] for more details).

In [116, 118] the following theorem is proved ('two-lines theorem'): if S is formally integrable, regular and ℓ-normal, then the terms $E_1^{p,q}$ of the $\mathcal{C}$-spectral sequence on $S^{(\infty)}$ with $p > 0$, $1 \leq q \leq n-2$ are trivial. In other words, non-trivial terms of the $\mathcal{C}$-spectral sequence are distributed on the column $E_i^{0,q}$ for $1 \leq q \leq n-2$ and on the rows $E_j^{p,n-1}$, $E_j^{p,n}$ for $p \geq 1$; this explains the name of the theorem. Note that $E_\infty^{0,q} = E_1^{0,q}$ for $1 \leq q \leq n-2$ and $E_\infty^{p,n-1} = E_3^{p,n-1}$, $E_\infty^{p,n} = E_3^{p,n}$ for $p \geq 1$. An explicit description of the non-vanishing terms is also provided by the two-line theorem.

Most 'classical' differential equations of mathematical physics (KdV equation, heat equation, etc.) are ℓ-normal, but gauge equations (like Yang-Mills equation and Einstein equation) are not; the structure of their conservation laws is more complex than that of ℓ-normal equations [51]. This fact was not considered in [116, 118]. In [107] the method of compatibility complex was proposed to compute the number of non-trivial lines. That approach has been generalized in [114] (k-lines theorem) and compared with the Koszul–Tate resolution method in [113]. In [22] the same problem was considered in the framework of exterior differential systems (the author used the term 'characteristic cohomology' to indicate what we called the horizontal de Rham cohomology); see also [23].

Since then, several papers dealt with the $\mathcal{C}$-spectral sequence on differential equations. We recall the works [50, 59] on evolution equations and the works [7, 8, 9] on second-order parabolic and hyperbolic equations in the case $n = 2$.

6.4 Variational sequence and symmetries

Invariant variational problems. There are a number of variational problems which admit a group of symmetries G. The way to find invariant solutions for these problems is to find solutions of a reduced system on the space of invariants of G; this is related to Palais'principle of symmetric criticality. In the paper [6] the solution of this problem is related to the existence of a cochain map between the G-invariant variational bicomplex (see below) and the variational bicomplex on the space of invariants of G. The local existence of the cochain map is related to a relative Lie algebra cohomology group.

Lie derivatives of variational forms. The Lie derivative of variational forms, *i.e.*, elements of $E_1^{p,n}$ or equivalently $\mathcal{V}^p$, is interesting for the determination of symmetries of Lagrangians and source forms. However, the result of a Lie derivative with respect to a prolonged vector field is a form which, in general, contains d_H-exact terms. For this reason it is natural to derive a new operator, the variational Lie derivative, which is defined up to d_H-exact terms. Such a formula first appeared in [118] ('infinitesimal Stokes'formula').

Theorem 6.4. *Let $X \colon E \to TE$ be a vector field, and $[\alpha] \in E_1^{p,n}$. Then*

$$[\mathcal{L}_{X^\infty}\alpha] = e_1([i_{X^\infty}\alpha]) + i_{X^\infty}(e_1([\alpha])),$$

where the contraction $i_{X^\infty}(e_1([\alpha]))$ is defined by virtue of the identity $i_{X_V^\infty} \circ d_H = d_H \circ i_{X_V^\infty}$ and the fact that the action of X_H^∞ is trivial.

The above theorem can also be found in [4], and in [38, 77] in the finite order case. It has clear connections with Noether's theorem, for which we invite the reader to consult the above literature.

Evolutionary vector fields are one example of first-order differential operators with no constant term that preserve the contact distribution. For this reason, they yield operators on all the spaces $E_k^{*,*}$ of the $\mathcal{C}$-spectral sequence. More generally, the problem of finding 'secondary' differential operators, *i.e.*, higher order differential operators which preserve the contact distribution, has been faced [53]. A complete classification has not been achieved yet.

Takens'problem. It is well-known that, by virtue of Noether's theorem, any infinitesimal symmetry of a Lagrangian yields a conservation law of the corresponding Euler–Lagrange equations. Takens'problem [101] can be formulated as follows: *when a source form, endowed with a space of infinitesimal symmetries each of which generates a conservation law, is locally variational.*

The problem has been solved in several cases, besides the simplest ones in [101].

1. Among the main results of [13], we have the following one. Consider the bundle $A \to M$, where A is the space of electromagnetic vector potentials, and let Δ be a source form. Suppose that Δ has translational and gauge symmetries and corresponding conservation laws. Then, if $n = 2$ and Δ is of third order, or $n \geq 3$ and Δ is of second order, Δ is locally variational.

2. In [10] the case of second-order scalar differential equations is considered. A number of conditions on symmetries and conservation laws about which Takens'problem for the above equations admits an affirmative answer is derived.

3. In [12] the case of polynomial differential equations which admit the algebra of Euclidean isometries and corresponding conservation laws is considered. The authors make use of the formal differential calculus by [42].

4. Finally, in [92] the problem is considered for the case of systems of first order differential equations which admit the group of translations and corresponding conservation laws.

Invariant inverse problem. This problem can be described as follows: given a locally variational source forms which is invariant under the action of a group G, find (if it exists) a Lagrangian which is invariant under the action of G.

The problem admits a formulation in cohomological terms: consider a Lie group G (or a Lie pseudogroup G') acting on a manifold E[8]. Lift the action to $J^\infty(E, n)$. Then consider the G-invariant subcomplex of the variational sequence. Its cohomology is the G-invariant cohomology; it determines the solvability of the invariant inverse problem. The main difference with the non-invariant case is that the G-invariant cohomology could be different from zero even locally. The same consideration holds for infinitesimal actions.

The invariant variational bicomplex appeared in [106] together with several examples of applications, but without any specific mention to the invariant inverse problem. In a subsequent paper [3] (where the reader can also find a short story of the invariant inverse problem) the following invariant inverse problem was considered: to find natural Lagrangians for natural source forms on the bundle of Riemannian metrics on a given manifold M. Among the results it is interesting to note that, while the invariant $n+1$-st cohomology vanishes for $\dim M = 0,\ 1,\ 2 \bmod 4$, it is nonvanishing for $\dim M = 3 \bmod$

[8]if the manifold is fibred, then the action is required to be projectable

4, thus leading to an obstruction of Chern–Simons type to the existence of natural Lagrangians for natural source forms.

Further results in mechanics ($n = 1$) are exposed in [85, 86], where the obstruction to the existence of Lagrangians is found in the cohomology of the Lie algebra of G. It is proved that such an obstruction can be removed by a central extension of the group G.

In [11] the local inverse problem invariant with respect to a finite-dimensional Lie group action is completely solved. Namely, conditions under which the local invariant cohomology of the variational sequence is isomorphic to the local invariant de Rham cohomology of the total space E are given. Moreover, considering the action of a finite-dimensional Lie algebra, conditions under which the local invariant cohomology of the variational sequence is isomorphic to the cohomology of the Lie algebra are given. The paper is completed by several examples. In [93] the case of an infinite-dimensional Lie pseudogroup has been considered, and the local invariant cohomology is computed in terms of the Lie algebra cohomology of the formal infinitesimal generators of the pseudogroup. An application of the above methods and results is presented in [94].

In [60] the authors make use of a method of invariantization from the moving frames theory and compute the invariant counterparts of operators like the horizontal differential and the Euler–Lagrange operator.

The invariant variational bicomplex seems to be an important part of the BRST theory of quantized gauge fields [16], despite the fact that the mathematical side of that theory still needs deep investigation.

Differential invariants. The works [3, 106] (see also references therein) showed that the $\mathcal{C}$-spectral sequence invariant with respect to the pseudogroup of local diffeomorphisms provides a new approach to characteristic classes. In [52, 120] characteristic classes are interpreted as cohomologies of the regular spectra of the algebra of differential invariants.

6.5 Further topics

Variational multivectors. Variational forms, *i.e.*, elements of $E_1^{p,n}$, admit a dual counterpart. More precisely, 'standard' differential forms on a manifold M admit as a counterpart multivector fields, *i.e.*, sections of the bundle $\wedge^k TM$. The counterpart of the 'standard' exterior differential is the Schouten bracket. The counterpart for variational forms is constituted by variational multivectors. In [91] (where the word 'functional' is used instead of 'variational', see also references therein) the approach to variational multivectors is an 'integral' one, and multivectors are described in coordinates up to total divergences. A variational Poisson bracket is introduced. In [56] multivectors are explicitly described through the calculus of differential operators, and their bracket is analyzed in the graded case, which leads both to a variational Poisson bracket and to a variational Schouten bracket. We stress that such a bracket allows to define operators which are Hamiltonian, in the sense that their 'squared' bracket vanishes, without respect to a given Hamiltonian [58].

Variational sequences on supermanifolds. The problem of computing the analogue of the $\mathcal{C}$-spectral sequence for supermanifolds is almost completely open. We quote the paper [112] with a comprehensive list of references. There, integration, adjoint operators, Green's formula, the Euler operator and Noether's theorem are introduced in a noncommutative setting. As a by-product, an interesting characterization of Berezin volume forms is obtained.

7 Notes on the development of the subject

To the author's knowledge, the first papers where a variational sequence appeared are by Horndeski [55] and by Gel′fand and Dikii [42]. Horndeski constructed an analogue of the sequence (29) for a class of tensors (rather than forms) in coordinates, using jets in an implicit way, in order to study the inverse problem of the calculus of variations. Gel′fand and Dikii introduced the differentials d_H and $\mathcal{E}$ of the variational sequence (22) in the case $n = m = 1$, only for polynomial functions of $u^i_{\boldsymbol{\sigma}}$. The calculus that they developed was called by them the *formal calculus of variations*. This calculus was used to study the Hamiltonian formalism for evolution equations, of which they are among the main contributors. Their variational sequence was studied by Olver and Shakiban, who computed its cohomology [89]. An alternative approach to this problem is in [31].

At the same time Tulczyjew, studying the Euler–Lagrange differential [108], and speaking with Horndeski[9], matured the ideas that led to the variational bicomplex, first for higher n-dimensional tangent bundles $T^r_n M$ [109], then for jets of fibrings [110]. His results included the local exactness of the variational bicomplex, achieved through local homotopy operators. However, his results did not include the solution of the global inverse problem, *i.e.*, cohomological results about the variational sequence, until [111].

The $\mathcal{C}$-spectral sequence approach was developed independently by Dedecker

[9] W. M. Tulczyjew, private communication

[29] and Vinogradov [115, 116]. However, the most complete achievements about the $\mathcal{C}$-spectral sequence are due to Vinogradov. In fact, in [29] there is only the definition of the $\mathcal{C}$-spectral sequence, together with the definition of the variational sequence on jets of fibrings and submanifolds (see also the later paper [30]). Previous works by Dedecker made use of spectral sequences for the calculus of variations [25, 26, 27, 28], but none dealt with variational sequences. In [115, 116] all terms of the $\mathcal{C}$-spectral sequence are computed for jets of fibrings and jets of submanifolds ('one-line theorem'). The computation included a complete description of all terms of $E_1^{p,n}$ through the theory of adjoint operators and Green's formula. Moreover, the $\mathcal{C}$-spectral sequence was computed for the first time also on differential equations ('two-line theorem'). This last achievement led to the interpretation of conservation laws in terms of cohomology classes of the horizontal de Rham complex on the given equation and their computation. Vinogradov did not publish the detailed proofs of his results in [115, 116]; however he published a longer exposition of his results in [117] followed by a detailed exposition with proofs in [118]. Manin's review [83] of the geometry of partial differential equations devotes a section to the variational sequence. The material is based on results by Vinogradov and Kuperschmidt.

Independently, Takens [102] provided a formulation of the variational bicomplex together with local exactness and global cohomological results on jets of fibrings. His proofs of the local exactness relied are different with respect to those of Tulczyjew. After [102], Takens left this field of research and become an outstanding scientist in dynamical systems.

All the above approaches to variational sequences were developed on infinite order jets. Independently from the previous authors, Anderson and Duchamp [5] developed a new approach to variational sequences. The main novelty in their approach was the use of finite order jets. Their approach was formulated trying to find spaces of forms for which d_H was stationary with respect to the order of jets. Their approach did not provide a 'long' variational sequence, stopping with zero just after the space of source forms. Moreover, in the paper there is a cohomological computation about the global inverse problem, but this is performed on the infinite order jet. Another important result in the paper is the local classification of trivial Lagrangians of order r (but see also [15] for Lagrangians which do not depend on (x^λ)). Such a result has never been derived in an infinite order jets framework. Anderson is the author of the book [4], which, unfortunately, has never been finished. However, it is still a source of interesting proofs, examples, and facts, especially about the finite order inverse problem.

After that the foundations were established, a number of important contributions and improvements appeared in the literature.

In [106] Tsujishita reviewed the $\mathcal{C}$-spectral sequence and presented some new proofs of old facts together with new ideas and theorems (remarkably, the invariant $\mathcal{C}$-spectral sequence with interesting examples). A deeper analysis by several authors (Gessler [51], Krasil′shchik [65], Marvan [80], Tsujishita [107], Verbovetsky [114]) led to the generalization of Vinogradov's 'two-lines theorem' to the so-called 'k-lines theorem'. The fundamental tool for the computation of non-trivial lines in the $\mathcal{C}$-spectral sequence was the compatibility complex (see [114] and references therein).

The k-lines theorem was also proved in [16] in the framework of the BRST theory of quantized gauge fields [16]. A comparison between the approach of [16] (Koszul-Tate resolution) and the compatibility complex method was recently performed [113].

Bryant and Griffiths [22] proved similar results on the horizontal de Rham cohomology in the framework of exterior differential systems. They call such a cohomology the *characteristic cohomology* of an exterior differential system.

Duzhin began to study the finite order $\mathcal{C}$-spectral sequence, but he only completed the computations for first order jets of the trivial bundle $\pi = \mathrm{pr}_1 \colon M \times \mathbb{R} \to M$ (here pr_1 is the projection on the first factor) [34].

Krupka was the first one to formulate a 'long' variational sequence on finite order jets in [70]. This approach was formulated in the language of sheaves entirely in terms of finite order jet spaces. The idea is described in section 5. The results included local exactness and global cohomology of the finite order variational sequence, which turned out to be the same as the infinite order case. More precisely, it was proved that the direct limit of Krupka's variational bicomplex was the same as the variational bicomplex [123, 124], and that the $\mathcal{C}$-spectral sequence on finite order jets provides a finite order variational sequence which is the same as Krupka's one [124, 125]. The representation of Krupka's variational sequence was obtained by Krbek and Musilová in [67, 68] using the interior Euler operator adapted to the finite order case. The classification of variationally trivial Lagrangians was proved using local exactness of the finite order variational sequence [48, 75]. The Lepagean equivalent theory provided yet another representation of the variational sequence [76].

As a final remark, we observe that there are many research topics which are connected with variational sequences (such as the inverse problem of the calculus of variations). It is impossible to provide historical notes for all of them, for space and time constraints. The interested reader can consult the references indicated in section 6.

Appendix: splitting the exterior algebra

In propositions 2.8 and 2.15 we deal with two splittings of exterior algebrae which are induced by the splittings (3) and (13) of the underlying space. In order to make this paper self-contained we briefly describe how to obtain the exterior algebra projections from the underlying splitting projections [122, 124].

Let V be a vector space such that $\dim V = n$. Suppose that $V = W_1 \oplus W_2$, with $p_1 : V \to W_1$ and $p_2 : V \to W_2$ the related projections. Then, we have the splitting

$$\wedge^m V = \bigoplus_{k+h=m} \wedge^k W_1 \wedge \wedge^h W_2, \tag{46}$$

where $\wedge^k W_1 \wedge \wedge^h W_2$ is the subspace of $\wedge^m V$ generated by the wedge products of elements of $\wedge^k W_1$ and $\wedge^h W_2$.

There exists a natural inclusion $\odot^k L(V,V) \subset L(\wedge^k V, \wedge^k V)$. Then, the following identity can be easily proved:

$$\odot^n (p_1 + p_2) = \sum_{i=0}^{n} \binom{n}{i} \odot^i p_1 \odot \odot^{n-i} p_2.$$

It follows that the projections $p_{k,h}$ related to the splitting (46) turn out to be the maps

$$p_{k,h} = \binom{k}{p} \odot^k p_1 \odot \odot^h p_2 : \wedge^m V \to \wedge^k W_1 \wedge \wedge^h W_2.$$

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